

From Stability to Chaos

A Hands-On Introduction to Nonlinear Dynamics

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1 A start at the end

1.1 Sensitive dependence on initial conditions: the butterfly effect

Around 1960 Edward Lorenz made an odd discovery. Working at MIT Lorenz worked on the issue of weather prediction, but rather than studying the full equations describing atmospheric flow and weather phenomena, he chose to focus on a highly simplified set of equations that still retained some essential elements of the atmospheric system. The governing equations of his abstract system are

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}\tag{1.1}$$

where r, b and σ are parameters and $\dot{x}$ denotes dx/dt . Making use of an early supercomputer he could numerically integrate the equations in time to obtain long-term predictions. In order to be able to calculate the system's behaviour over very long periods, Lorenz sometimes started a new simulation from the 'halfway' result of a calculation that was still running. But when he plotted such a continued calculation together with the original calculation in one graph, he noted to his utter surprise that after a brief period the two curves started to diverge rapidly, ultimately displaying entirely different behaviour. Lorenz figured out that the "stored" values that he used as initial conditions for the second calculations differed slightly from the original values due to round-off effects. This led him to conclude that a minute perturbation of the initial conditions can lead to enormous differences over time. The perspective of weather prediction provided an excellent metaphor to express the effect that small causes can have big impacts: *Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?*

With a modern PC we can today easily retrace Lorenz' footsteps and experience ourselves the remarkably rich behaviour of his simple system as well as the sensitive dependence on initial conditions. To this end, we start `maple` and load the file `LorenzSensitiveDependence_start.mws`, which contains an implementation of the system (1.1), listed below (text comments are preceded by an `'|'`)

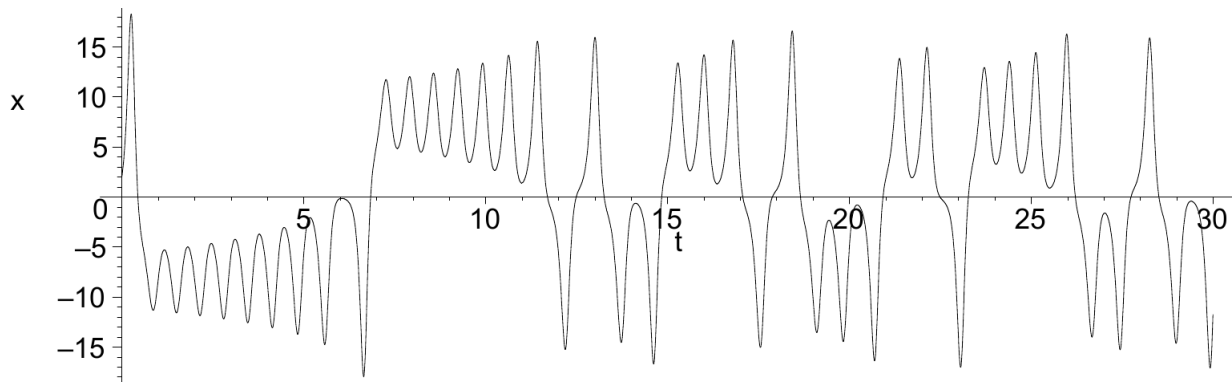


Figure 1.1: Time series $x(t)$ of the Lorenz system (1.1).



Maple

```
restart; with(plots): with(DEtools):
# Define the Lorenz equations
f1 := (x, y, z) -> sigma * (y - x);
f2 := (x, y, z) -> -x * z + r * x - y;
f3 := (x, y, z) -> x * y - b * z;
DE1 := diff(x(t), t) = f1(x(t), y(t), z(t));
DE2 := diff(y(t), t) = f2(x(t), y(t), z(t));
DE3 := diff(z(t), t) = f3(x(t), y(t), z(t));
DE := DE1, DE2, DE3;
# Set the parameters
r:=28; b:=8/3; sigma:=10;
# Set the initial conditions
ic1 := x(0)=2, y(0)=5, z(0)=5;
# Solve system with dsolve.
sol1 := dsolve({DE, ic1}, [x(t), y(t), z(t)], type=numeric, maxfun=-1);
# Set the integration range ..
tstart := 0; tend := 30;
# And plot the timeseries ..
odeplot(sol1, [t, x(t)], tstart..tend, labels=["t", "x"]);
```

Running the entire worksheet produces the time series of $x(t)$ shown in Figure 1.1.

The time series of $x(t)$ seems to oscillate peacefully, yet no regular pattern has established itself in the range $t \in [0, 30]$. One might suspect that this is merely the *transient* phase — the period before the system settles into an equilibrium behaviour — and extend the integration time by adjusting `tstart` and `tend` to, for example, `100...130` or `1000...1030`; plotting $y(t)$ or $z(t)$ instead of $x(t)$ tells the same story. However long the integration time is chosen, no regular pattern emerges. This is not yet a proof of chaos, since the transient might simply be even longer. As will be elaborated in the chapters to follow, chaotic behaviour is defined in terms of how sensitively the system responds to a small change in the initial conditions. We can probe this by integrating the Lorenz system for two slightly differing initial conditions: removing the `#` symbols at the end of the worksheet activates the commands below, which produce plots like Figure 1.2.

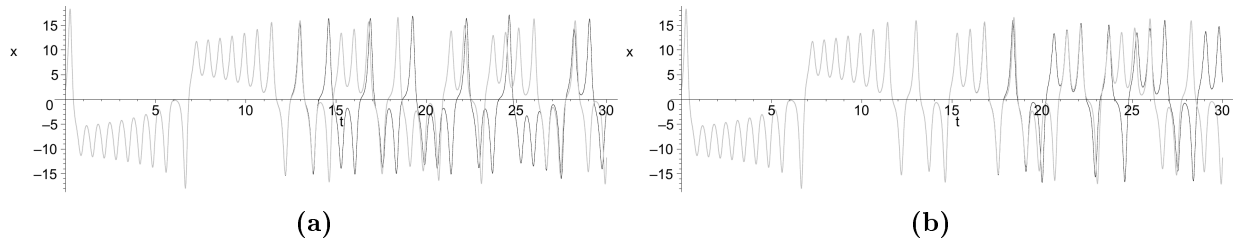


Figure 1.2: Sensitive dependence on initial conditions in the Lorenz-system. Two time-series with slightly differing initial conditions in x : $x(0) = 2$ and $x(0) = 2 + \epsilon$, respectively. In all cases $y(0) = 5, z(0) = 5$. (a) $\epsilon = 10^{-3}$. (b) $\epsilon = 10^{-5}$.



Maple

```
ic2 := x(0)=2+epsilon,y(0)=5,z(0)=5;
epsilon := 1E-3;
sol2 := dsolve({DE,ic2},[x(t),y(t),z(t)],type=numeric,maxfun=-1);
# plot 2 timeseries in one plot.
tstart := 0; tend := 30;
p1 := odeplot(sol1,[t,x(t)],tstart..tend,labels=["t","x"],color=blue);
p2 := odeplot(sol2,[t,x(t)],tstart..tend,labels=["t","x"],color=red);
display([p1,p2]);
```

Reducing ϵ delays the moment at which the two series visibly diverge, but they always do so eventually. If ϵ is made small enough the solutions appear to stay together indefinitely; this is deceptive, however, and reflects the finite accuracy of the computation rather than the dynamics of the system. Increasing the working precision — for instance with `> Digits := 20;`, which instructs `maple` to use 20 digits instead of the default 10 — restores the divergence.

Apart from analysing the time-series, much can be learned by studying the system in the 3d-*phase-space*. In `maple` this requires only a minor change of the plot command: `odeplot(sol1,[x(t),y(t),z(t)],tstart..tend,labels=["x","y","z"]);` which gives a plot like [Figure 1.3](#).

Viewed from various angles, the phase-space trajectory reveals the celebrated butterfly shape. Trajectories launched from different initial conditions — even one as far away as $x(0) = 100$ — are all drawn onto the same object: they converge upon the *attractor* (compare [Figure 1.3](#)).

See the file `LorenzSensitiveDependence.mws` for a worked-out maple-sheet. The Lorenz system will be studied in more detail in chapter 6.

1.2 Celestial Uncertainties

Imagine three celestial objects – like stars, planets, moons, meteorites and/or satellites – which influence each other by their gravitational pull. Would it be possible to fully predict the future trajectories of these three interacting objects?

It may come as a surprise that the answer turns out to be 'no, not in general'. As discussed during the first lecture, this so-called 'three body problem' has a long history involving among others Euler, Laplace and Poincaré. In the end it was Henri Poincaré who managed to prove that the system cannot be solved analytically in general situations. Only in a few special cases can the solution be expressed in a closed form. To get a feeling for the enormous complexity of the three body problem, we will analyse the system below and calculate numerically some possible

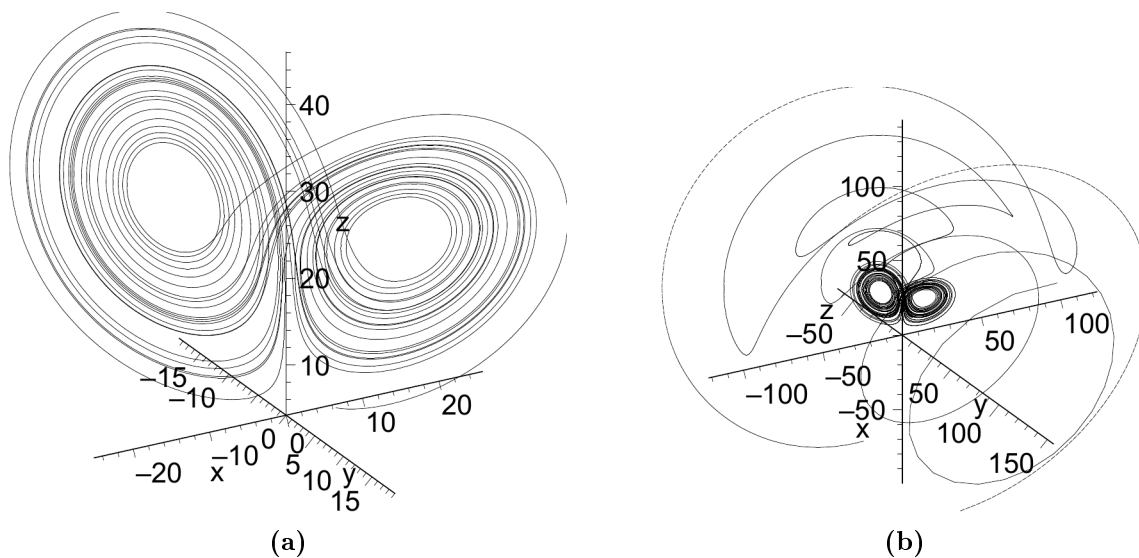


Figure 1.3: Trajectories in phase-space. (a) 'butterfly'-shaped appearance of the system dynamics in phase space. (b) 'attractor'; as the trajectories converge on the butterfly, the object appears to attract the trajectories.

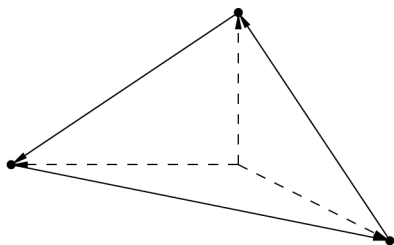


Figure 1.4: Three-body problem.

trajectories with the aid of `maple`.

Consider Figure 1.4. The magnitude of the gravitational force of m_2 on m_1 is given by Newton's law

$$f_{21} = G \frac{m_1 m_2}{|\mathbf{x}_2 - \mathbf{x}_1|^2}$$

with G the gravitational constant given in Table 1.1. The direction of this force can be expressed by $(\mathbf{x}_2 - \mathbf{x}_1)/|\mathbf{x}_2 - \mathbf{x}_1|$. Hence the complete three body problem can be written as

$$\begin{aligned} m_1 \ddot{\mathbf{x}}_1 &= G m_1 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3} + G m_1 m_3 \frac{\mathbf{x}_3 - \mathbf{x}_1}{|\mathbf{x}_3 - \mathbf{x}_1|^3} \\ m_2 \ddot{\mathbf{x}}_2 &= G m_2 m_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} + G m_2 m_3 \frac{\mathbf{x}_3 - \mathbf{x}_2}{|\mathbf{x}_3 - \mathbf{x}_2|^3} \\ m_3 \ddot{\mathbf{x}}_3 &= G m_3 m_1 \frac{\mathbf{x}_1 - \mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_3|^3} + G m_3 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_3}{|\mathbf{x}_2 - \mathbf{x}_3|^3} \end{aligned} \quad (1.2)$$

The system constitutes a phase-space with dimension 18: three variables each with six degrees of freedom (three-dimensional position and velocity).

This eighteen-dimensional problem can be reduced dramatically by exploiting the particular structure of the earth-moon-satellite setting. Because the satellite (m_3) is exceedingly light compared with the earth (m_1) and the moon (m_2), i.e. $m_3 \ll m_1, m_2$, it is pulled around by the two heavy bodies but exerts no appreciable influence on them. If, in addition, the two heavy bodies (the *primaries*) are taken to move on circular orbits about their common centre of mass, and the satellite is confined to that orbital plane, the eighteen dimensions collapse to just four. This reduction, from Newton's law (1.2) to the compact system below, is carried out step by step in Appendix A. Writing the satellite position as $\mathbf{x}_3(t) = (x(t), y(t))$ and its velocity components as $u = \dot{x}$, $v = \dot{y}$, the resulting equations of motion are

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= G m_1 [x_1 - x]/d_1^3 + G m_2 [x_2 - x]/d_2^3 \\ \dot{v} &= G m_1 [y_1 - y]/d_1^3 + G m_2 [y_2 - y]/d_2^3 \end{aligned} \quad (1.3)$$

with d_1 and d_2 the distances from the satellite to the earth and to the moon,

$$d_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2} \quad d_2 = \sqrt{(x_2 - x)^2 + (y_2 - y)^2} \quad (1.4)$$

and the earth- and moon-positions moving on their circular orbits

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = -\frac{m_2 R}{m_1 + m_2} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{m_1 R}{m_1 + m_2} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix} \quad (1.5)$$

Here R is the (constant) earth-moon distance and ω the angular frequency of the circular orbit. The parameter values for the problem are listed in Table 1.1. Based on these values one finds for the angular frequency $\omega = 2.96 \cdot 10^{-6} \text{ rad s}^{-1}$.

Table 1.1: Parameters in the three-body problem.

gravitational constant	G	$6.67 \cdot 10^{-11}$	$\text{Nm}^2 \text{kg}^{-1}$
distance between earth and moon	R	$3.84 \cdot 10^8$	m
mass of earth	m_1	$5.97 \cdot 10^{24}$	kg
mass of 'moon' in example	m_2	$m_1/4$	kg
real mass of the moon		$7.36 \cdot 10^{22}$	kg

With these substantial simplifications the order of the system has been reduced to four and does not look overly complicated, but it still cannot be analytically solved. With use of **maple** we can however integrate the system numerically and study the complex trajectories that occur for different initial positions and velocities of the satellite. To simplify the numerical integration a bit, we take for the mass of the moon $m_2 = m_1/4$ instead of the real value $m_2 = m_1/81$. Examples of trajectories for different initial conditions are given in Figure 1.5. One observes regular trajectories (orbits around the Earth or the moon) and irregular (chaotic) trajectories. Additional insight can be gained by viewing the trajectories in a reference frame that co-rotates with the circular orbit. In this view both the earth and moon have a stationary position, which simplifies the interpretation of the satellite trajectory.

A characteristic feature of chaos is the sensitive dependence on initial conditions. This feature can be nicely observed in Figure 1.6 in which two trajectories are plotted for slightly different initial conditions (relative difference 0.01%).

In the exercise you can manipulate a **maple** -worksheet that contains an implementation of the three-body problem (1.3). You can change the initial position and velocity of the satellite and get a feeling for the sensitivity on the initial conditions.

1.3 Exercises

1.3.1 Sensitive dependence on initial conditions in the Lorenz System

Downloads: LorenzSensitiveDependence_start.mws, LorenzSensitiveDependence.mws

Study the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}\tag{1.6}$$

with the parameters given $r = 28, b = 8/3, \sigma = 10$.

Login on Blackboard/ChaoticProcesses, go to Course Documents/01 and save the file LorenzSensitiveDependence_start.mws on your computer. Start Maple Classic and load the file.

- Execute the entire maple file once by clicking on the !!!*-button. Have a look at the time series of $x(t)$.
- Plot the time series of $y(t)$ instead of $x(t)$. (Note: you only need to slightly modify the command `odeplot`). Study also $z(t)$.
- Change the time range by modifying `tstart` and `tend`.

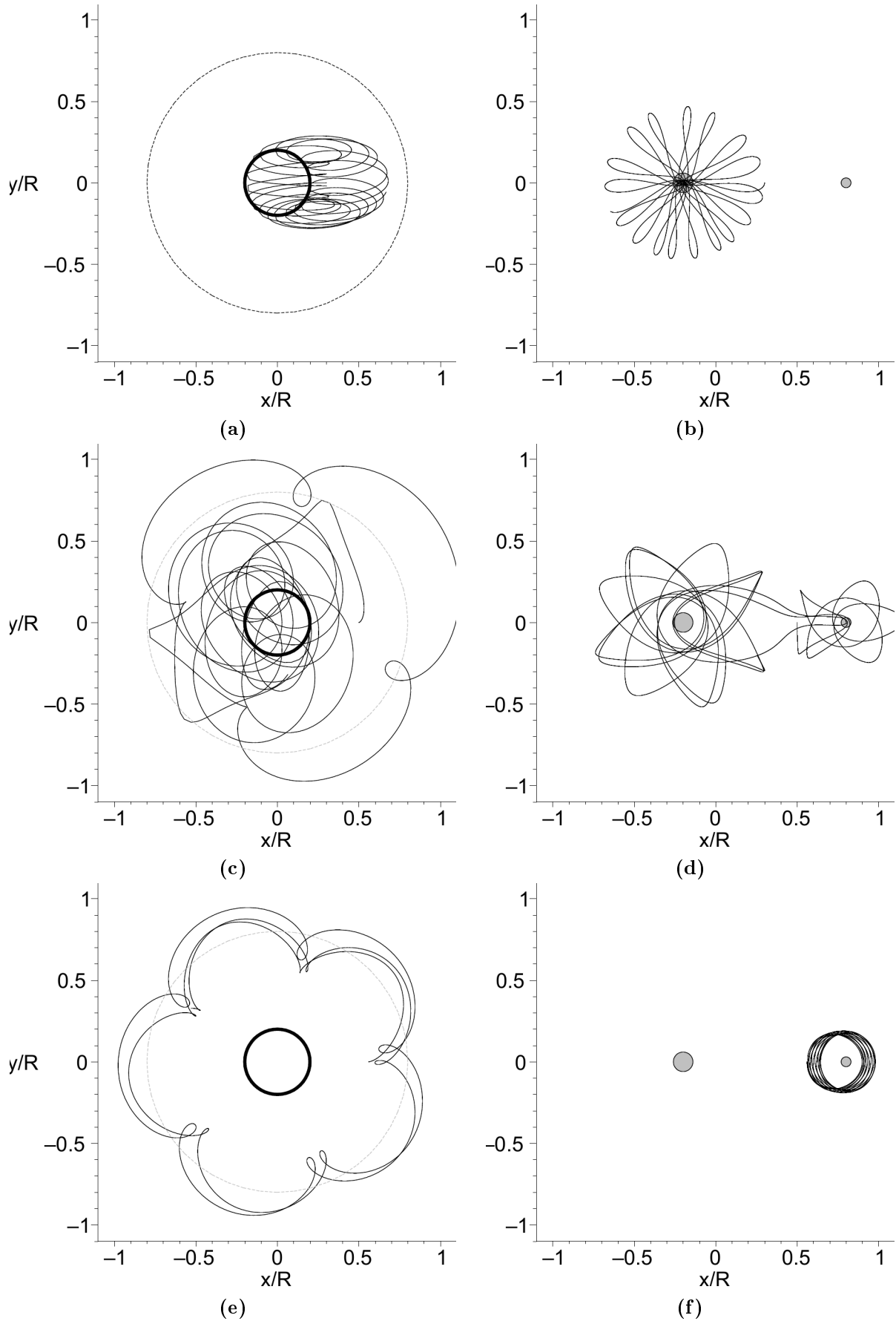


Figure 1.5: Some examples of possible trajectories in the three-body problem for different initial conditions. In all cases $y(0) = u(0) = v(0) = 0$; $x(0)$ is being varied. The integration time is two months. Left row: Trajectories of the earth (thick black line), moon (gray line) and satellite (thin black line), plotted in the reference frame that moves with the center of mass. Right row: Same situation as left but trajectories plotted in a reference frame that additionally co-rotates with the earth and moon. a,b] Satellite orbits around the earth. c,d] Satellite has a chaotic

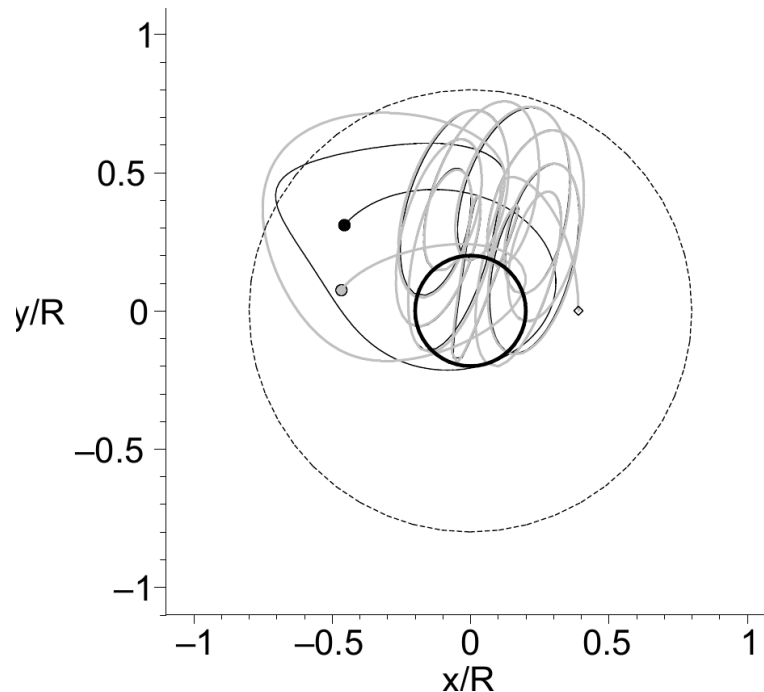


Figure 1.6: Sensitivity on initial conditions: two trajectories initiated from slightly different positions with a relative difference in $x(0)$ of 10^{-4} . Integration time is two months. Diamond: initial locations. Circles: end position of satellites after two months. Top: trajectories in reference frame that moves with the center of mass. Bottom: the trajectories in the reference frame that additionally co-rotates with the earth and moon.

d) Delete the comment symbols ``#'` at the end of the file. Now you can study simultaneously the system's behaviour for two different initial conditions. The difference between the initial values of x is ϵ . Reduce the value of ϵ and observe the effect.

e) Reduce ϵ until the solutions are no longer different. Increase `tend` to make sure. Can you understand this seemingly critical value of ϵ ?

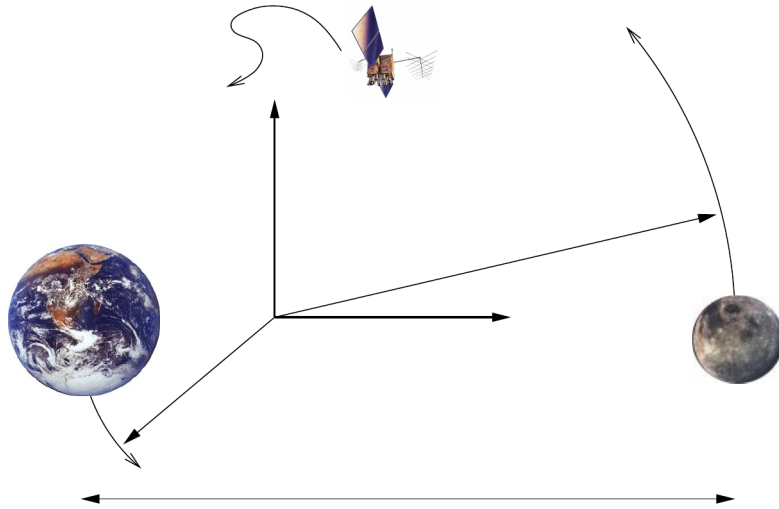
f) Make a plot of the 3d-*phase-space* by plotting $[x(t), y(t), z(t)]$ instead of $[t, x(t)]$. Again you can do this with a minor change of the `odeplot`-command. You may want to increase `numpoints` as well, to get a smoother 3d-graph.

g) If everything went well you saw a so-called “strange attractor”. Why it is called *strange* will be dealt with later, but to see that the object is really attracting, change the initial condition $x(0) = 2$ to $x(0) = 100$ and look at the trajectories in phase-space. Study other initial conditions and verify that the trajectories converge upon the *attractor*.

1.3.2 Classification of timeseries

Downloads: `ClassifyingTimeseries.mws`, `SomeDatasets.mws`

Open the Maple file `SomeDatasets.mws` to view different data sets `set1`, `set2`, `set3`. Find out which one is a: pure random noise, b: a one dimensional mapping, c: a higher order mapping. Explain your answer.



1.3.3 Three body problem

Downloads: ThreeBody.mws, ThreeBody_start.mws, ex0036r03_fig1.pstex

Consider a satellite in space under influence of the gravitational pull of the earth and the moon.

gravitational constant	G	$6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-1}$
distance between earth and moon	R	$3.84 \times 10^8 \text{ m}$
mass of earth	m_1	$5.97 \times 10^{24} \text{ kg}$
mass of moon	m_2	$7.36 \times 10^{22} \text{ kg}$

For simplicity, the orbits of earth and moon are assumed to be circular. Defining the position of the earth and moon as $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, and their mean distance as R , the motion is parametrized as

$$\mathbf{x}_1 = -r_1 \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \quad \mathbf{x}_2 = r_2 \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix},$$

$$r_1 = R \frac{m_2}{m_1 + m_2}, \quad r_2 = R \frac{m_1}{m_1 + m_2}, \quad \omega = 2\sqrt{\frac{G(m_1 + m_2)}{R^3}},$$

where r_1 and r_2 are the radii of earth and moon, and ω denotes the angular frequency. Using the law of gravitation $F = G \frac{mm'}{r^2}$, the motion of the satellite is governed by

$$\ddot{\mathbf{x}} = Gm_1 \frac{\mathbf{x}_1 - \mathbf{x}}{|\mathbf{x}_1 - \mathbf{x}|^3} + Gm_2 \frac{\mathbf{x}_2 - \mathbf{x}}{|\mathbf{x}_2 - \mathbf{x}|^3}.$$

Remarkably, this relatively simple system has very complex and rich behavior. The worksheet **ThreeBody_start.mws** contains an implementation of this system and to magnify the chaotic behavior, we have increased the mass of the moon to 25% of the earth's mass.

a) Vary the satellite's initial position x_0 in the range $[-5 \cdot 10^8, 5 \cdot 10^8]$ and determine for which interval the system behaves chaotic. Study also the system in a co-rotating reference frame.

b) Check the sensitive dependence on initial conditions. Take $x_0 = 2 \times 10^8 \text{ m}$ and set the perturbation $\epsilon = 1 \text{ m}$. How many days does it take before there is a visual difference between the satellite's original and perturbed trajectory?

c) In reality the mass of the moon is a factor 81 lower than the earth's. If you change this, are there still chaotic trajectories? You may need to increase the initial velocity a bit.

1.3.4 Attractor reconstruction

Downloads: `LorenzReconstruct.mws`, `LorenzReconstructStart.mws`, `LorenzConstruct.mws`, `dataset1.txt`, `dataset2.txt`

a) Study the data set `dataset1.txt`, which consists of a one-dimensional time series $xx[1] \dots xx[N]$, “measured” with a constant sampling rate (i.e. with a constant time interval). Try to reconstruct the attractor by making a three-dimensional plot of the points $(xx[j], xx[j + \Delta j], xx[j + 2\Delta j])$, for all possible j . The maple command `spacecurve` may be useful in this respect. See also the file `LorenzReconstructStart.mws`. Change Δj to get the best result.

b) Do the same for `dataset2.txt`, which has some additional noise superimposed.

2 Chaos in Iterative Maps

In the previous chapter we saw that a relatively simple system like the 3-body problem can exhibit chaotic behaviour. The trajectories in phase-space appear irregular to us, with no convergence to some sort of systematic, repetitive, pattern. In addition we saw that the computed solutions depended strongly on the initial conditions. A slight perturbation of the initial conditions, however small, gave rise to trajectories that sooner or later had nothing to do with the trajectories of the unperturbed system. This is the essence of chaotic systems. Because in any physical system one cannot know exactly the initial conditions, chaotic systems are in essence unpredictable in the long run. How long it takes for the system to reach this so-called 'prediction-horizon' depends 1) on the accuracy with which one knows the initial conditions, 2) on the accuracy of the computer on which the calculation is performed and 3) on the intrinsic dynamics of the system itself. In the next chapters we aim to make these notions more precise.

While the governing equations of the system in chapter 1 were not overly complicated – in striking contrast with the complexity of the system's behaviour, we will see in this chapter that the essential aspects of chaos can be found in systems that are even more elementary: these are the so-called discrete maps, or iterative maps. Because the governing equations are so simple, discrete systems form an excellent entry to chaos from which we can benefit a lot when encountering more complex systems.

A one-dimensional iterative map has the form

$$x_{n+1} = f(x_n) \quad n = 0, 1, 2, 3, \dots \quad (2.1)$$

Starting with an initial value x_0 one obtains a series by repeatedly applying (2.1), i.e. $x_1 = f(x_0)$, $x_2 = f(x_1)$, etc. As an example consider the map $f(x) = rx$ that is sometimes used as a simple model for bacterial growth:

$$x_{n+1} = rx_n \quad (2.2)$$

In this equation x then represents the (normalised) number of bacteria at generation n and r represents the growth rate. The evolution of the time series for any r and x_0 can be predicted without problem. Working forward $x_1 = rx_0$, $x_2 = rx_1 = r^2x_0$, $x_3 = r^3x_0, \dots$, we find that x_n will depend on r and x_0 via

$$x_n = r^n x_0 \quad n = 0, 1, 2, \dots \quad (2.3)$$

The long term behaviour depends critically on r ; for $n \rightarrow \infty$, $x_n \rightarrow 0$ if $|r| < 1$, whereas $x_n/x_0 \rightarrow \infty$ for $|r| > 1$; only for $r = 1$ nothing happens, $x_n = x_0$.

The behaviour of this *linear* mapping is clearly not very exciting. As a model for bacterial growth it is also rather unrealistic to the extent that unbounded growth cannot persist owing to an

inevitable shortage of food and mutual competition. These effects can be taken into account by adding an extra factor $(1 - x_n)$ to the growth model. As soon as the (normalised) population x nears 1, this factor then reduces the effective growth rate. Thus taking $f(x) = rx(1 - x)$ in (2.1) one obtains the so-called Verhulst-model [May, 1976], also referred to as the logistic map:

$$x_{n+1} = rx_n(1 - x_n) \quad r \in [0, 4] \quad x_0 \in [0, 1] \quad (2.4)$$

The crucial aspect that makes the behaviour of (2.4) so interesting resides in the *non-linearity* of $f(x)$ – the map has a quadratic term in x . Due to this non-linearity it is very hard to find an analytical solution for x_n in terms of r and x_0 , as could be done for the linear mapping (2.3). Instead let us resort to a numerical approach and write a small **maple** program to calculate and plot the series for some values of r . Take for example $r = 3.9$ and $0 < x_0 < 1$. Execute the **maple** program and observe the chaos! Now try other values of r , like $r = 3.4$, $r = 2.5$, and $r = 0.5$, and execute the program again. In Figure 2.1 we have plotted the series for four different values of r . Note the fundamentally different behaviour in each case. One observes stationary behaviour, periodic behaviour and chaotic behaviour.



```
restart; with(plots):
f := x -> r * x * (1 - x);
generate time series
N := 20;
X := Array(0..N):
r := 0.5;
X[0] := 0.1;
for n from 0 to N-1 do
    X[n+1] := evalf( f(X[n]) );
end do;
plot time series
l := seq([n, X[n]], n=0..N):
pointplot(l, labels=["n", "xn"], symbolsize=10);
```

One could ask why r has to be confined between $[0, 4]$. The reason is that in this range all x_n will be bounded between $[0, 1]$ provided that also the initial condition $x_0 \in [0, 1]$. This follows from the fact that the maximum of $f(x)$ is at $x = 1/2$ regardless of the value of r . The maximum value is therefore $f(1/2) = r/4$, while the minimum value of f in the interval $[0, 1]$ is always 0. So, when $x_n \in [0, 1]$, we know that $x_{n+1} \in [0, r/4]$, which explains that r must be within $[0, 4]$.

2.1 Graphical analysis of maps

2.1.1 The Cobweb method

An interesting graphical way to see how discrete maps work is by making so-called cob-web plots. It works as follows: Plot the mapping function $y = f(x)$ together with the line $y = x$, as is done for $r = 2.7$ in Figure 2.2. Now starting at x_0 use the graph $y = f(x)$ to find the value of x_1 by following the solid line in Figure 2.2. The trick to proceed is to use the line $y = x$, i.e. follow the horizontal line to the diagonal, then follow the vertical dotted line giving you the point x_1 on the x -axis. Repeating the previous steps you find graphically $x_2 = f(x_1)$, then $x_3 = f(x_2)$ and so on, until you find a fixed point as in Figure 2.2, or for example a period-2 solution as in Figure 2.2 or chaos as shown in Figure 2.2. Note that in the latter two plots we refrained from plotting the dotted lines, because, as one soon realizes, one does not need to really go back to the x - and

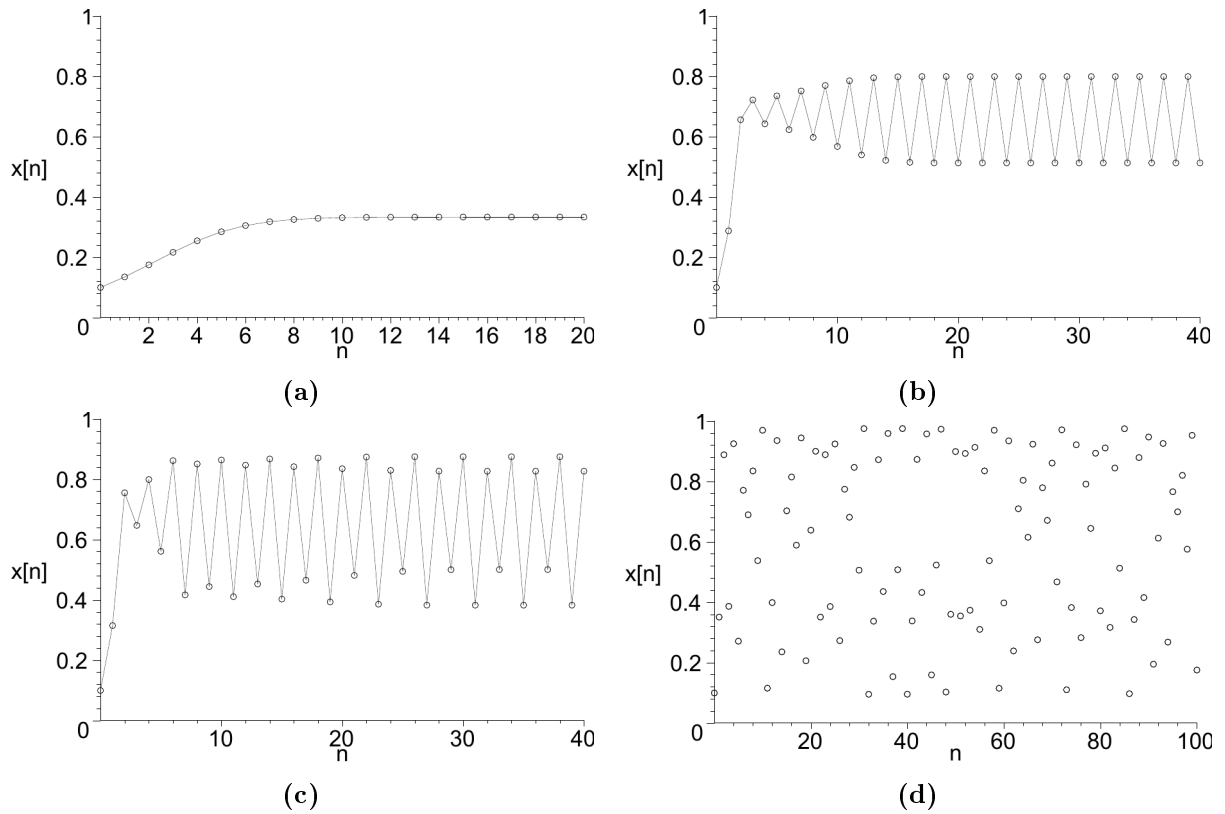


Figure 2.1: The rich behaviour of the logistic map (2.4) for various values of the parameter r . (a) $r = 1.5$ convergence to a fixed point. (b) $r = 3.2$, convergence to a period-2 solution. (c) $r = 3.5$, convergence to a period-4 solution. (d) $r = 3.9$, chaos.

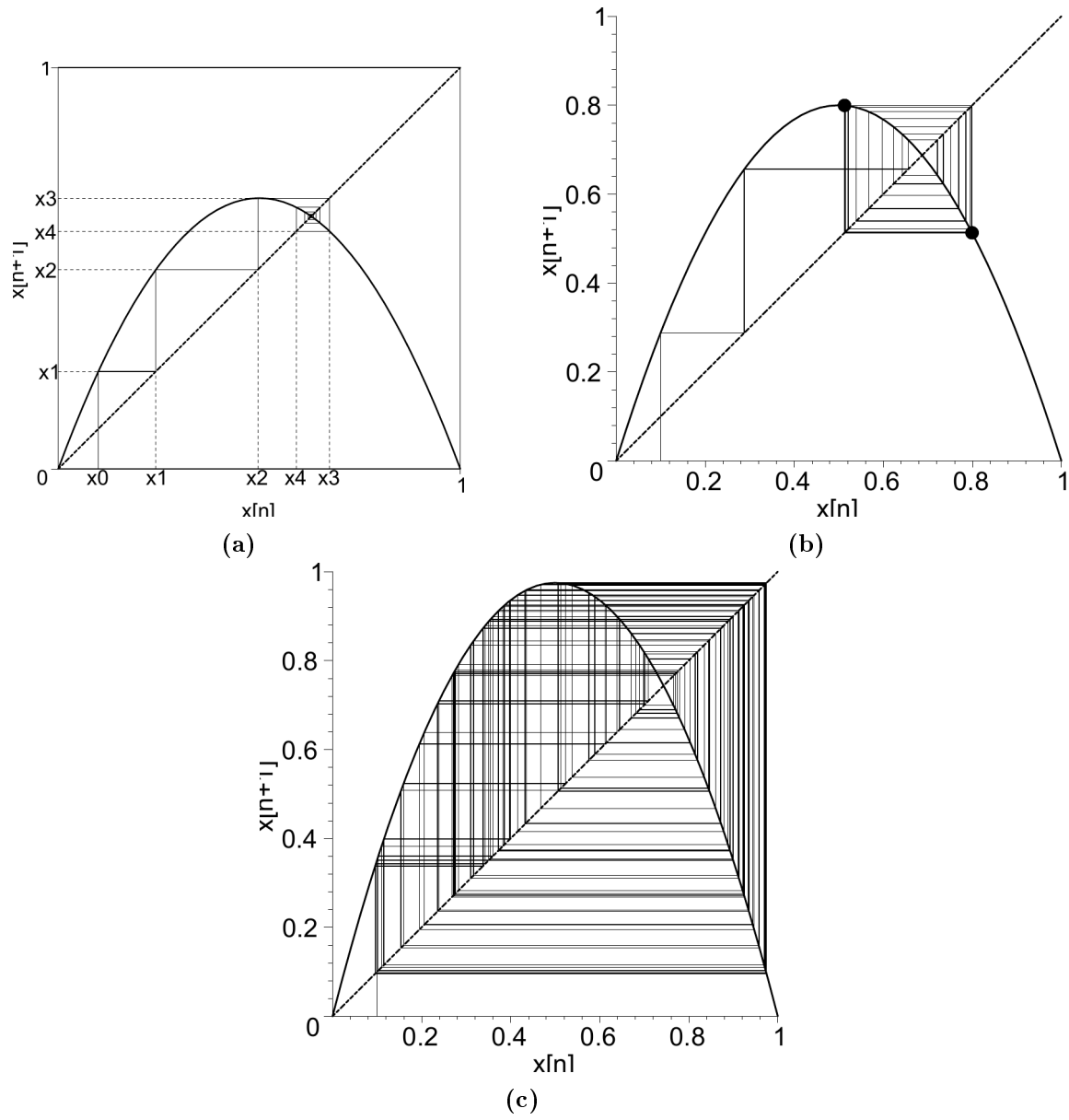


Figure 2.2: Cobweb graphs of the logistic map for various r . (a) $r = 2.7$ convergence to a fixed point. (b) $r = 3.2$, convergence to a period-2 solution. (c) $r = 3.9$, chaos.

y -axis for finding the next point. Instead, it suffices to move back and forth between the graphs of $y = f(x)$ and $y = x$.

One can study the cobweb method in more detail by running the **maple** program below and changing the value of r and/or looking at other mappings like $f(x) = r \sin(\pi x)$.



Maple

```
restart; with(plots):
f := x -> r * x * (1 - x);
N := 20;
X := Array(0..N):
r := 3.9;
X[0] := 0.1;
for n from 0 to N-1 do
  X[n+1] := evalf( f(X[n]) );
end do;
# Create and plot the cobweb
pp := [X[0], 0]:
for n from 0 to N-1 do
  pp := pp, [X[n], X[n]], [X[n], X[n+1]];
end do;
plot([f(x), x, [pp] ], x=0..1,0..1, color = [black, blue, red],
      labels=["x[n]", "x[n+1]"]);
```

2.1.2 The Return plot

Another useful method to study the outcome of discrete maps is by making a so-called *return-plot*, a plot where one plots the value of x_{n+1} versus the previous value x_n for consecutive n . Usually it makes sense to take n larger than some number in order to disregard the transient behaviour, i.e. the initial part of the time series in which the system is still evolving to its equilibrium state (which could be periodic or chaotic). As an example we plot in Figure 2.3 the return-plots corresponding to the time-series of Figure 2.1. Here is the **maple** -code to make a return-plot.



Maple

```
restart; with(plots):
f := x -> r * x * (1 - x);
N := 200; X := Array(0..N):
r := 3.9; X[0] := 0.1;
for n from 0 to N-1 do
  X[n+1] := evalf( f(X[n]) );
end do;
nmin := 50;
xr := seq([X[n], X[n+1]], n=nmin..N-1):
pointplot([xr], labels=["x[n]", "x[n+1]"], view=[0..1,0..1]);
```

The information conveyed by the return-plots may seem slightly trivial when one knows the expression of the mapping that was used to generate the data, as is the case for the example of Figure 2.3 where it was easy to plot the gray line representing the mapping $f(x)$. But in a practical situation where one analyses the results of some experiment one usually does not know exactly which mapping describes the data-sets. In such a situation it is useful to make a return-plot since it gives an impression what kind of mapping lurks behind. Consider for example the

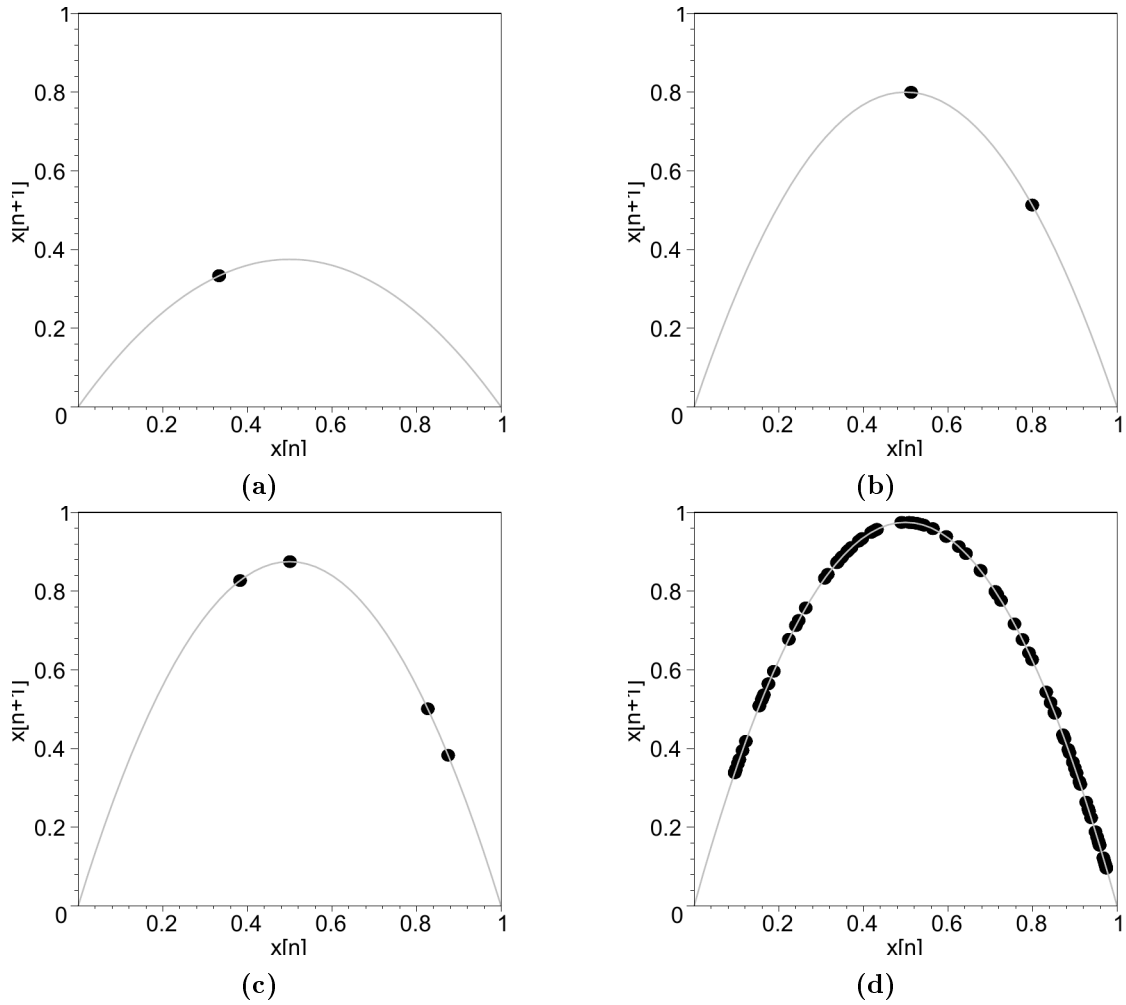


Figure 2.3: Return-plots of the logistic-map data corresponding to the time series plotted in Figure 2.1. Solid gray line represents the mapping (2.4) (a) $r = 1.5$, fixed point. (b) $r = 3.2$, period-2 solution. (c) $r = 3.5$, period-4 solution. (d) $r = 3.9$, chaos.

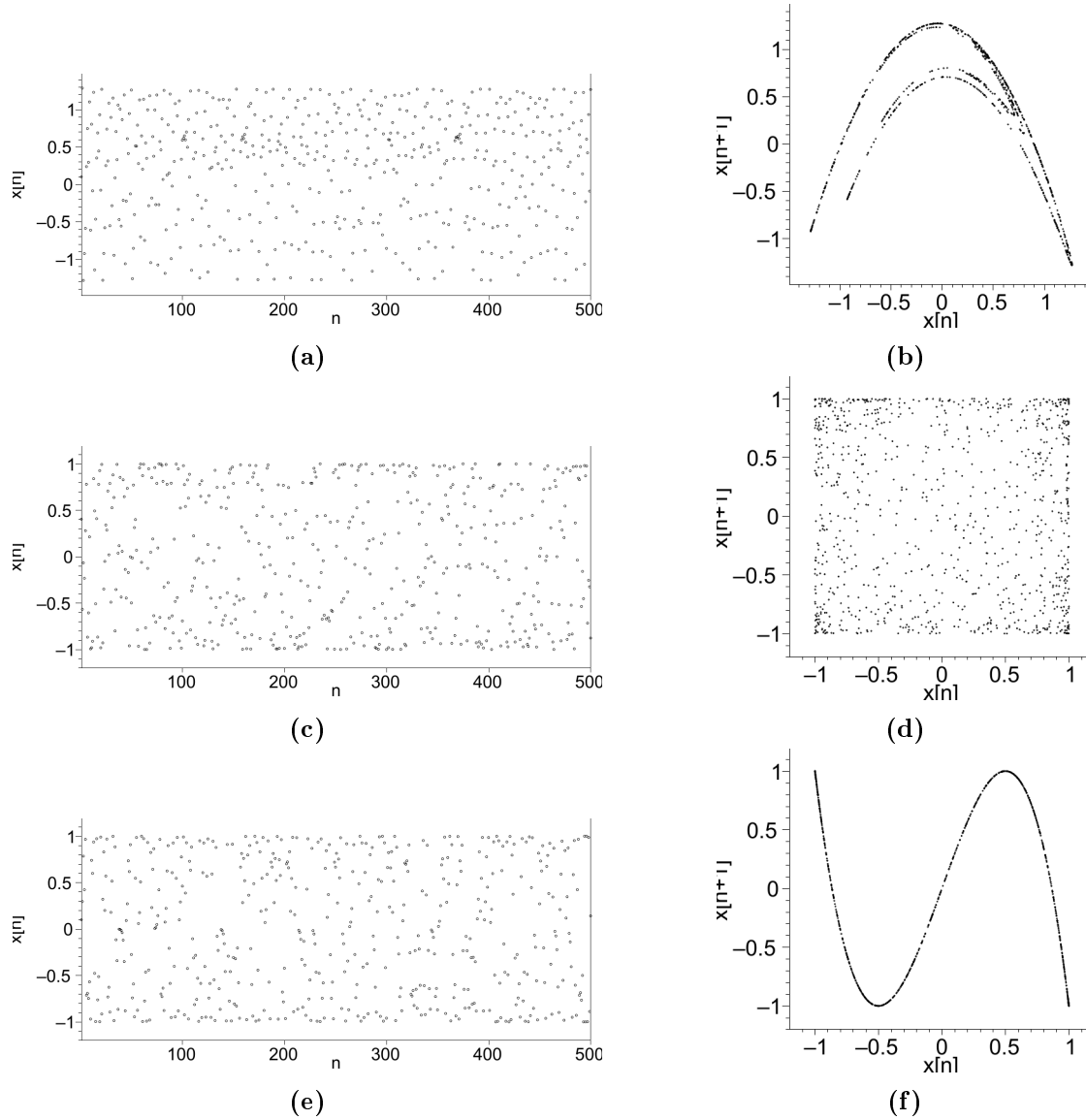


Figure 2.4: Three (unknown) data-sets: their time series and corresponding return-plots.

three time-series plotted in the left column in Figure 2.4. Judging from the time-series alone, one has a hard time figuring out which data-set is 1) uncorrelated noise, 2) chaotic data from a one-dimensional mapping, 3) chaotic data from a higher-dimensional mapping. The return-plots in the right column, on the other hand, are very instructive in this respect: clearly the first time-series originates from a higher dimensional mapping because the data-points in the return-plot cannot be described by a function of x alone; the second time-series is the uncorrelated noise since the return-plot does not display any correlation between x_n and x_{n+1} ; the third time-series is well described by a first-order mapping (it is $x_{n+1} = -4x_n^3 + 3x_n$).

2.2 Fixed points and stability analysis

While gaining some understanding of chaotic behaviour is the main objective of this chapter, it will prove useful to first get some firm ground under our feet and focus on the stationary behaviour we noticed in the system for $r < 3$, i.e. when the series converges to a fixed point $x_n \rightarrow x^*$, see for example Figure 2.1. A fixed point solution of the map $x_{n+1} = f(x_n)$ can be found analytically by realizing that in the end: $\lim_{n \rightarrow \infty} x_{n+1} = x_n$, hence the fixed point solution x^* to (2.1) must satisfy

$$x^* = f(x^*) \quad (2.5)$$

For the logistic map we find two possibilities

$$x^* = rx^*(1 - x^*) \quad \rightarrow \quad x^* = 0, \text{ or } x^* = 1 - \frac{1}{r} \quad (2.6)$$

Indeed, the series for $r = 3/2$ depicted in Figure 2.1 converges to $x_n \rightarrow x^* = 1/3$ as predicted by (2.6). It will do that for any initial value $x \in (0, 1)$ as shown in Figure 2.5. The other fixed point of (2.6), $x^* = 0$, is only attainable when you start at exactly $x_0 = 0$ or $x_0 = 1$. Any deviation will inevitably lead to $x^* = 1/3$. So we can say that for $r = 3/2$ there is a fixed point $x^* = 1/3$ which is *stable*, and a fixed point $x^* = 0$ which is *unstable*.

Let us try to make the notion of stability of fixed points more precise. If we start the sequence at a fixed point $x_0 = x^*$, the system will remain in the fixed point by definition, see (2.5). But what if we slightly perturb the initial value, i.e. we start with $x_0 = x^* + \epsilon_0$, where the perturbation is really small: $\epsilon_0 \ll 1$. Will the series x_n return to the fixed point? Or will it diverge? Put differently, if we write $x_n = x^* + \epsilon_n$, then will the deviations ϵ_n approach to zero for increasing n or not?

One can answer this question by substituting $x_n = x^* + \epsilon_n$ in (2.2) and employing a Taylor expansion of f around x^*

$$x^* + \epsilon_{n+1} = f(x^* + \epsilon_n) = f(x^*) + \epsilon_n f'(x^*) + \mathcal{O}(\epsilon_n^2) \quad (2.7)$$

where $f'(x^*)$ is the shorthand notation for $\left. \frac{df}{dx} \right|_{x=x^*}$. Using the fixed point definition $x^* = f(x^*)$ and neglecting the higher order terms in ϵ one arrives at the following evolution equation (or mapping) for the deviations ϵ_n :

$$\epsilon_{n+1} = \epsilon_n \sigma \quad (2.8)$$

where $\sigma = f'(x^*)$. It is important to note that σ does not depend on n . The ϵ series (2.8) is therefore similar to the series produced by the linear equation (2.2) studied earlier. Using (2.3) we can express ϵ_n in terms of the initial perturbation ϵ_0 and the derivative of f at the fixed point

$$\epsilon_n = \epsilon_0 \sigma^n \quad (2.9)$$

Clearly the value of σ determines the stability of the system, since $\epsilon_n \rightarrow 0$ when $-1 < \sigma < 1$. We conclude

$$\sigma = f'(x^*) \quad \begin{cases} |\sigma| > 1 & x^* \text{ is an unstable fixed point} \\ |\sigma| < 1 & x^* \text{ is a stable fixed point} \end{cases} \quad (2.10)$$

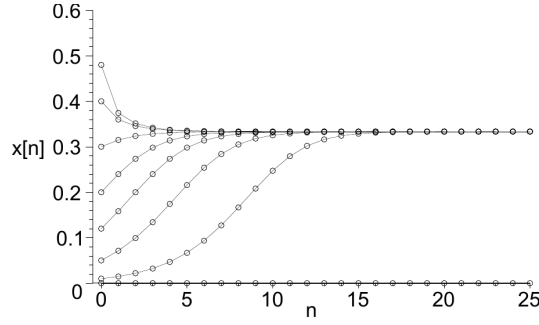


Figure 2.5: Series of the logistic map at $r = 1.5$ for different initial values x_0 . The fixed point $x^* = 1/3$ is clearly stable. The fixed point $x^* = 0$ is unstable since a small perturbation is sufficient to escape from it.

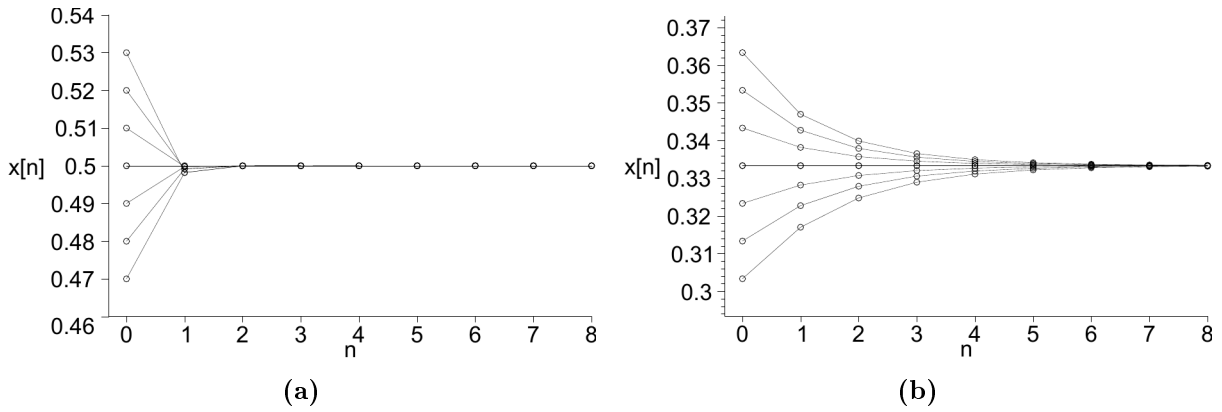


Figure 2.6: The convergence to the fixed point in a superstable situation and an ordinary stable situation. (a) $r = 2$ Superstable situation, i.e. $\sigma = 0$: the convergence to the fixed point is extremely fast. (b) $r = 1.5$ local convergence to the fixed point $x^* = 1/3$.

If the derivative is positive and smaller than 1, i.e. $0 < \sigma < 1$, then the system will monotonically approach the fixed point. If $-1 < \sigma < 0$, the system will converge to the fixed point x^* in an alternating fashion. Note that when $|f'(x^*)| = 1$ exactly, we cannot decide on the stability of the fixed point on the basis of the first derivative f' alone. In such a situation (marginal stability) we must have a closer look at the higher order terms appearing in (2.7), as the governing equation for the deviations becomes

$$\epsilon_{n+1} = \epsilon_n + \frac{1}{2}f''(x^*)\epsilon_n^2 + \mathcal{O}(\epsilon_n^3)$$

In this book we will generally not deal with these kind of exceptions.

A situation of special interest though, is when $\sigma = 0$. In this case the system is called *superstable*, since the convergence to the fixed point is extremely fast as one can observe in Figure 2.6. Clearly the convergence rate is much faster than the exponential behaviour described by (2.9) for $\sigma \neq 0$. To find the convergence behaviour in the superstable case we must retain the second-order term in (2.7). If we write $a = f''(x^*)/2$, the deviations are governed by $\epsilon_{n+1} = a\epsilon_n^2$, which enables one to express ϵ_n as a function of ϵ_0 :

$$\epsilon_n = a^{2^n - 1} \epsilon_0^{2^n} \quad (2.11)$$

In (2.11) one should pay attention to the entirely different dependence on ϵ_0 compared to the linear dependence in (2.9), which explains the unusually rapid convergence in case of superstability.

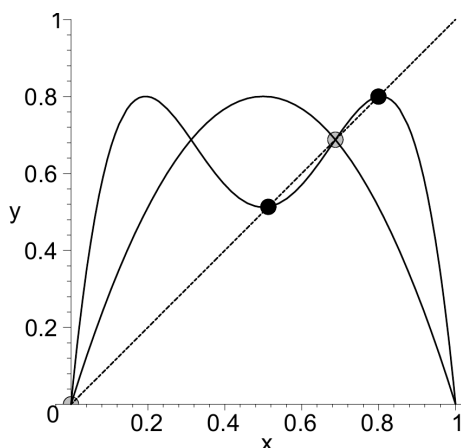


Figure 2.7: Plot of $f(x)$ and $g(x) = f(f(x))$ for $r = 3.2$, together with $y = x$. The period-2 solution, which satisfies $x = g(x)$, is indicated with the solid black circles. The two (unstable) fixed points that correspond to period-1 solutions, i.e. $x = f(x)$, are indicated by gray circles. Note that these points also satisfy $x = g(x)$ and could be mistakenly interpreted as period-2 solutions.



Example

The fixed points for the logistic map were given in (2.6): $x^* = 0$, $x^* = 1 - \frac{1}{r}$. The derivative is $f'(x) = r(1 - 2x)$. Inserting the first fixed point yields

$$f'(0) = r$$

(2.10) tells us that $x^* = 0$ is stable for $0 \leq r < 1$. Similarly we have

$$f'(1 - \frac{1}{r}) = 2 - r$$

which shows that the fixed point $x^* = 1 - 1/r$ is stable when $|2 - r| < 1$, i.e. when $r \in (1, 3)$. The fixed point is superstable at $r = 2$.

2.3 Periodic solutions and their stability

In the time-series plots of the logistic map shown in Figure 2.1 we observe, apart from chaos and stationary solutions, sustained periodic behaviour (viz. Figure 2.1 and Figure 2.1). Below we will show how one can find periodic solutions in discrete maps and how one can determine the stability of these solutions. We start with a period-2 solution and then generalize the method for periodic solutions of arbitrary order.

A period-2 solution, as for example depicted in Figure 2.1, is a solution that looks like

$$x_1^*, x_2^*, x_1^*, x_2^*, x_1^*, x_2^*, \dots$$

Apparently these points map onto each other and satisfy the following relation

$$x_2^* = f(x_1^*) \quad x_1^* = f(x_2^*) \quad (2.12)$$

One could also state that for n large enough period-2 solutions satisfy $x_{n+2} = f(x_n)$. Both

considerations lead to

$$x_1^* = f(f(x_1^*)) \quad x_2^* = f(f(x_2^*)) \quad (2.13)$$

which shows that both x_1^* and x_2^* are fixed points to the map

$$x^* = g(x^*) \quad g(x) = f(f(x)) \quad (2.14)$$

For the logistic map we have for example

$$g(x) = r f(x)(1 - f(x)) = r^2 x(1 - x)(1 - rx + rx^2) \quad (2.15)$$

which is plotted for $r = 3.2$ in Figure 2.7, together with $y = x$, and $y = f(x)$. The period-2 solutions (2.14) are the crossing points of the curves $y = g(x)$ and $y = x$ and can be determined analytically or numerically by solving $x = g(x)$. But it is important to bear in mind that a solution to $x = f(x)$ is also a solution to $x = g(x)$. Indeed, the curves $f(x)$ and $g(x)$ cross each other in the fixed point solution indicated by the gray dot in Figure 2.7. This means that if one has found solutions to $x = g(x)$, one has to be careful not to interpret these solutions necessarily as period-2 solutions, because it could also just be a period-1 solution.



Example

In the following maple program we determine the true period-2 solutions for the logistic map.

```
restart; with(plots):
f := x->r*x*(1-x);
g := x->f(f(x));
fp2 := {solve(x=g(x),x)};
```

maple returns the general solution to $x = g(x)$, where g is given by (2.15); we have added the curly bars to store the solution in the form of a set:

$$\text{fp2} = \left\{ 0, 1 - \frac{1}{r}, \frac{r+1 - \sqrt{r^2 - 2r - 3}}{2r}, \frac{r+1 + \sqrt{r^2 - 2r - 3}}{2r} \right\}$$

In this set we also recognize the fixed points solutions $x^* = f(x^*)$ derived earlier in (2.6). To get the 'true' period-2 solutions, we calculate the period-1 solutions separately and exclude this set of solutions from the former set by issuing the maple command `minus`:

```
fp1 := {solve(x=f(x),x)};
truefp2 := fp2 minus fp1;
```

$$\text{truefp2} = \left\{ \frac{r+1 \pm \sqrt{r^2 - 2r - 3}}{2r} \right\} \quad (2.16)$$

Substituting $r = 3.2$ we get $x_{1,2}^* \approx \{0.513, 0.799\}$, consistent with Figure 2.1.

2.3.1 Stability of period-2 solutions

Now that we have seen that period-2 solutions to $f(x)$ are simply fixed-point solutions to the map $g(x) = f(f(x))$, we can get directly derive their stability from (2.10):

$$\sigma_1^{(2)} = g'(x_1^*) \quad \text{and} \quad \sigma_2^{(2)} = g'(x_2^*)$$

where the superscript (2) reminds us that we are dealing with a period-2 solution. Although not *a priori* clear, it turns out that both expressions yield the same value for the stability, i.e. $\sigma_1^{(2)} = \sigma_2^{(2)}$. To see this, we work out the derivative using the chain-rule for differentiation

$$\begin{aligned}\sigma_1^{(2)} &= \frac{d}{dx} [f(f(x))]_{x=x_1^*} = [f'(f(x)) f'(x)]_{x=x_1^*} \\ &= f'(f(x_1^*)) f'(x_1^*) = f'(x_2^*) f'(x_1^*)\end{aligned}$$

where in the last step we used (2.12). In the same fashion one can work out the expression for $\sigma_2^{(2)}$ leading to the same outcome. So, we can drop the subscripts and state for the stability

$$\sigma^{(2)} = g'(x_1^*) = g'(x_2^*) = f'(x_1^*) f'(x_2^*) \quad (2.17)$$

As in (2.10), the period-2 solution pair (x_1^*, x_2^*) is stable if $\sigma^{(2)}$ computed from (2.17) satisfies

$$|\sigma^{(2)}| < 1$$



Example

To determine the stability range of the period-2 solutions of the logistic map we extend the maple-code of the previous example by the following commands

```
dg := D(g):
dg(x);
```

This gives

$$g'(x) = r^2(1 - 2rx + 2rx^2)(2x - 1)$$

Next we determine the stability range by

```
solve(abs(dg(truefp2[1]))=1,r);
```

which, ignoring negative solutions, yields

$$3, 1 + \sqrt{6}$$

Hence the stability range for the period-2 solution is $r \in \langle 3, 3.4494 \rangle$. The period-2 solution is found to be superstable (i.e. $\sigma^{(2)} = 0$) for $r = 1 + \sqrt{5} = 3.236 \dots$

2.3.2 Higher order periodic solutions and their stability

The notions discussed above can easily be generalized to periodic solutions of any order. A periodic solution of order m , or shortly a period- m solution, looks like

$$x_1^*, x_2^*, \dots, x_m^*, x_1^*, x_2^*, \dots, x_m^*, x_1^*, x_2^*, \dots, x_m^*, \dots$$

which are related to each other by the following relations

$$x_1^* = f(x_m^*), \quad x_2^* = f(x_1^*), \quad x_3^* = f(x_2^*), \quad \dots \quad (2.18)$$

One can also state that for n large enough, a period- m solution satisfies $x_{n+m} = f(x_n)$. Hence we define $g(x) = f^{(m)}(x) = f(f(f(\dots f(x))))$, i.e. g is equal to m -times recursive application of f . The equation from which the periodic solutions can be derived is then

$$x_k^* = g(x_k^*) \quad \text{all } k = 1, \dots, m \quad (2.19)$$

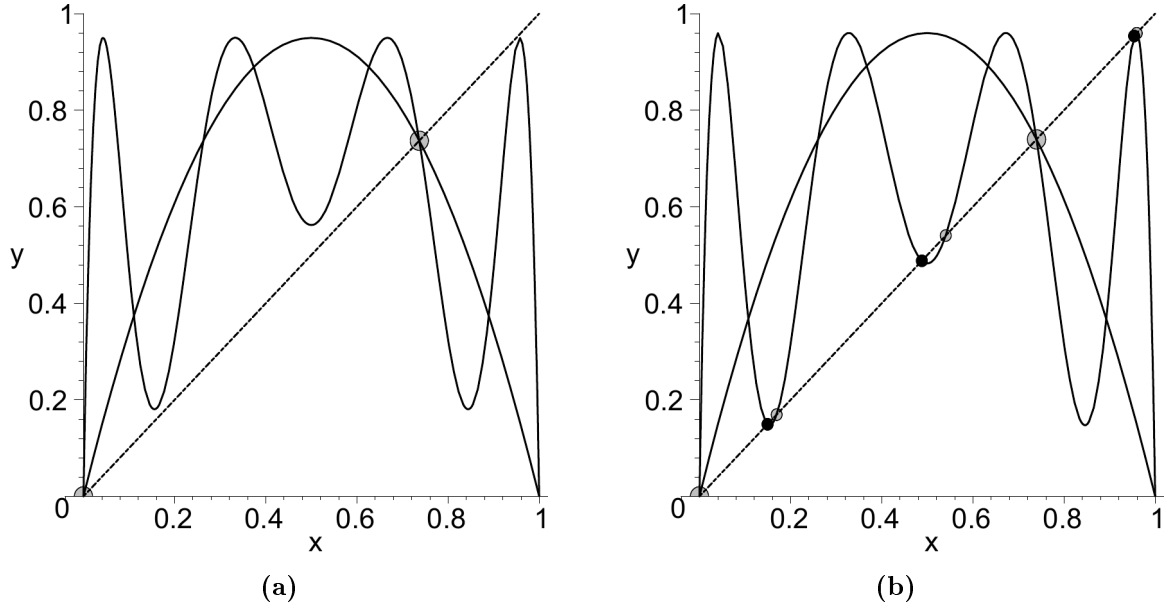


Figure 2.8: Plots of $f(x)$ and $g(x) = f^3(x)$ together with $y = x$. True period-3 solutions are indicated with the solid small black circles (stable) and small gray circles (unstable). The (unstable) fixed points $x = f(x)$ are indicated by big gray circles. Note that these points also satisfy $x = g(x)$. (a) $r = 3.80$, no true period-3 exists. (b) $r = 3.84$, two sets of period-3 solutions are found.

Again one must be careful with the interpretation of the solutions to (2.19), since they might contain 'stowaways', i.e. the solution of lower order periodic solutions. For example, when looking for a period-4 solution, not only the fixed point $x^* = f(x^*)$ will satisfy (2.19), but also the period-2 solution $x^* = f^{(2)}(x^*)$ will contaminate the outcome.

The stability of the period- m solution can be derived either from

$$\sigma^{(m)} = g'(x_1^*) = g'(x_2^*) = \dots = g'(x_m^*) \quad (2.20)$$

or, equivalently, from

$$\sigma^{(m)} = f'(x_1^*)f'(x_2^*) \dots f'(x_m^*) \quad (2.21)$$

The period- m solution is stable if $\sigma^{(m)}$ in (2.20) or (2.21) satisfies

$$|\sigma^{(m)}| < 1 \quad (2.22)$$

**Example**

To find a period-3 solution for the logistic map we must solve $x^* = g(x^*)$ with $g(x) = f^{(3)}(x)$. In Fig Figure 2.8 we have plotted $y = g(x)$ together with $y = f(x)$ and $y = x$. The left graph is for $r = 3.80$ and shows that there are no crossings between $y = x$ and $y = g(x)$ other than the crossings $x = f(x)$, implying that there are no period-3 solutions possible. The right graph is for $r = 3.84$. For this value there are *two* sets of period-3 solutions, i.e. there are six solutions, next to the fixed point solutions $x^* = f(x^*)$ (indicated by the big gray dots). One set of the period-3 solutions turns out to be stable (indicated by the black dots), whereas the other set appears to be unstable (indicated by the gray dots).

```
restart; with(plots):
f := x->r*x*(1-x);
g := x->(f@@3)(x);
fp1 := {solve(x=f(x),x)};
fp3 := {solve(x=g(x),x)};
truefp3 := fp3 minus fp1;
r := 3.84;
tfp3 := allvalues(op(truefp3));
```

The latter command yields

```
tfp3 := 0.149406, 0.169433, 0.488004, 0.540387, 0.953736, 0.959447
```

The stability can be calculated using (2.20). We determine the derivative g' by the `maple` command `D(g)`, which directly yields the derivative as a function. Next we use the `seq` command to enumerate all the values

```
dg := D(g):
seq(dg(tfp3[i]), i=1..6);
-0.875276, 2.74407, -0.875276, 2.74407, 2.74407, -0.875276
```

which confirms the message of (2.20) to the extent that the stabilities of the individual solutions x_k that constitute a periodic solution are the same.

An alternative way to calculate the stability is by using (2.21). We start with one solution and apply (2.18) to find the other values that belong to the same periodic solution. This is necessary since the expression for `tfp3` does not tell which solutions belong to each other.

```
..
df := D(f):
x3a := tfp3[1]: x3b := f(x3a): x3c := f(x3b):
stab := df(x3a)*df(x3b)*df(x3c);
      stab := -.875276

x3a := tfp3[2]: x3b := f(x3a): x3c := f(x3b):
stab := df(x3a)*df(x3b)*df(x3c);
      stab := 2.74407
```

As expected, the stability values match exactly those of the previous method.

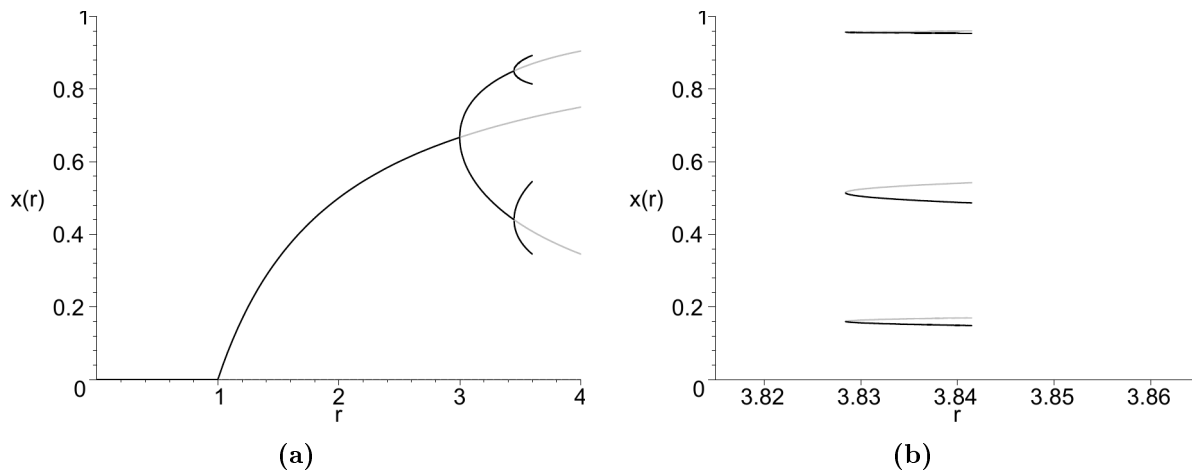


Figure 2.9: Bifurcation diagrams based on the analytical solutions to the logistic map (a) Period-doubling. (b) Period-3.

2.4 Bifurcation-diagrams

The fixed points solutions and periodic solutions usually depend on the parameters of the system. It is therefore often very useful to make a graph of the solutions as a function of the parameter(s) of the system. In Figure 2.9 we have done this for the logistic map, plotting as a function of the parameter r the fixed point solution (2.6), the period-2 solution (2.16), the period-4 solution, and, in the right graph, the period-3 solution. The left graph nicely shows that the fixed-point loses its stability at the exact same point where the period-2 solution becomes stable. It seems as if the solution is split into two branches. This is called a bifurcation, after the Latin word *bifurc* meaning fork. Then for larger r the period-2 solution becomes unstable itself and bifurcates into a period-4 solution. Although not shown here, the period-4 solution soon bifurcates again into a period-8 solution, and so on. This process of subsequent bifurcations is called *period-doubling*. We will get back to this issue in section 2.8.

In Figure 2.9 the solutions for a period-3 are plotted. It turns out that this solution cannot exist for r -values below $3.828427\dots$. In the graph one observes the stable and unstable branch, the existence of which we already encountered in Figure 2.8. But the stable branch (black lines) becomes unstable quite soon and bifurcates (not shown) into a stable period-6 solution for $r > 3.841499\dots$. The reason for not showing the period-6 solution is that the analysis becomes rather involved. With use of `maple` it is not impossible but surely it is rather cumbersome.

We mention in passing that `maple` does provide a fast *numerical* way to get a quick glance of the bifurcations of a map. See the small code below where we calculate and plot the solutions to $x = f^{(4)}(x)$ for the *sine map*

$$x_{n+1} = f(x_n) = r \sin(\pi x_n)$$

We do this by employing the `maple` command `implicitplot`.

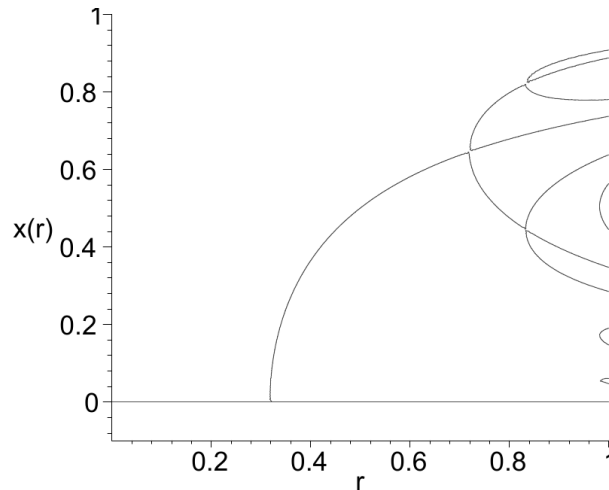


Figure 2.10: Numerical solutions to $x = f^{(4)}(x)$ as a function of $r \in [0.1]$ for the sine map $x_{n+1} = f(x_n) = r \sin(\pi x_n)$ calculated and plotted by using the `maple` command `implicitplot`



Maple

```
restart; with(plots):
f := x->r*sin(Pi*x);
implicitplot([x=(f@@4)(x)],r=0..1,x=0..1,grid=[150,150],
  view=[0..1,-0.1..1],labels=["r","x(r)"],thickness=2,axes=framed);
```

The method cannot distinguish between true period-4 solutions and double period-2 solutions, or quadruple period-1 solutions. Nor does it provide insight into the stability of the solutions. Nevertheless the method is capable of rapidly providing the global structure of the solutions. Note the extra period-4 solutions near $r = 1$. They are not stable however.

An alternative efficient method to study the equilibrium behaviour as a function of the system parameters is the following. The idea is to keep r fixed and iterate the map under study for, say, 1000 steps and then plot only the *last* half of the series at that particular r -value. By throwing away the first part of the series one disregards on purpose the *transient* behaviour, allowing one to focus on the *equilibrium* behaviour of the system. If one repeats this procedure for different values of r one obtains a plot like Figure 2.11, usually referred to as the 'bifurcation diagram' of the map.

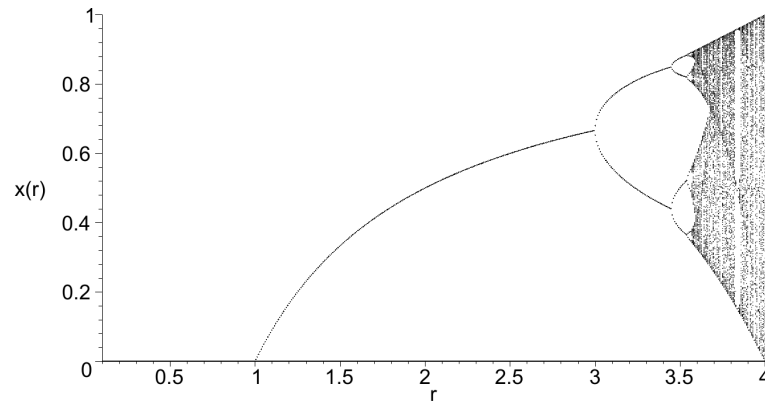


Figure 2.11: Bifurcation diagram of the logistic map obtained by plotting the entire series for different values of r

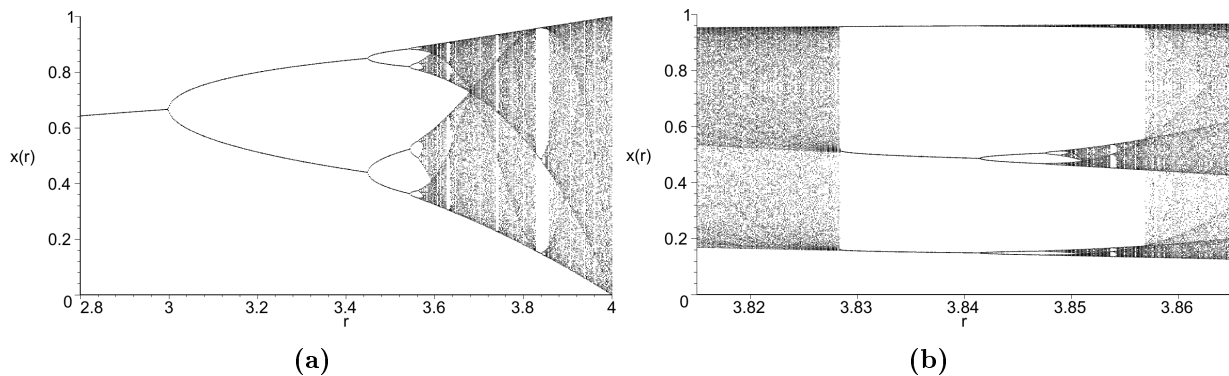


Figure 2.12: Various close-ups of Figure 2.11 (a) Period doubling route to chaos. (b) Region around the period-3 solution.



Maple

```
restart; with(plots):
f := x -> r * x * (1 - x);
Nr := 500; rmin := 3.3; rmax := 4;
N := 1000; Nmin := ceil(0.8*N);
X := Array(0..N):
X[0] := 0.57:
for i from 0 to Nr do
  r := evalf(rmin + (rmax - rmin)*i/Nr);
  for n from 0 to N-1 do X[n+1] := evalf( f(X[n]) ); end do:
  pl[i] := seq([r,X[n]],n=Nmin..N);
end do:
pointplot([seq(pl[i],i=1..Nr)],symbolsize=1,view=[rmin..rmax,0..1],
  labels=["r", "x(r)"]);
```

This bifurcation diagram is very instructive to the extent that one can see in one blink for which values of r there is chaos, or periodic behaviour. By zooming in on a specific range of r , as done in Figure 2.12 one can get a closer look at the behaviour. One may have noted the correspondence between the bifurcation diagrams and the analytical graphs in Figure 2.9. The difference is that with the method just outlined one will not obtain the unstable branches appearing in Figure 2.9.

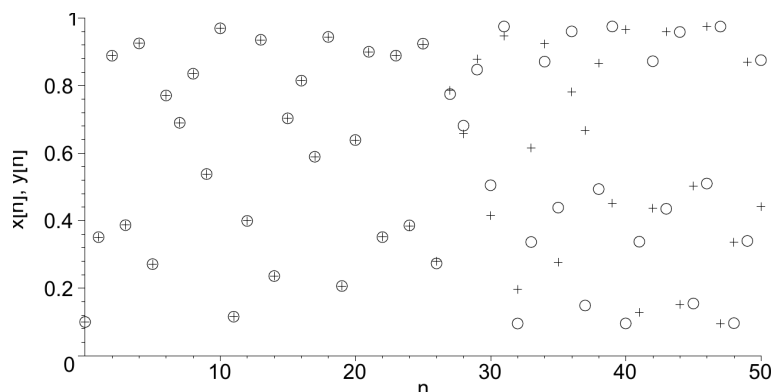


Figure 2.13: Logistic map at $r = 3.9$. Two sets with slightly different initial conditions $y_0 = x_0 + 10^{-9}$. The difference becomes already obvious at $n = 25$.

Closer inspection of the graphs in Figure 2.12 shows two 'routes to chaos'. In the left graph one observes the subsequent period-doubling for increasing r , which tend to happen sooner and sooner, until for $r = r_\infty \approx 3.56\dots$ the behaviour is no longer periodic but becomes chaotic. In the next section we make this notion more precise, how one can distinguish chaos from periodic behaviour with a very high order. Another way to enter chaos becomes clear from Figure 2.12: if one starts at $r = 3.83$ in the stable period-3 solution and one gradually *lowers* r then at $r = 3.82\dots$ the period-3 solution loses its stability and the behaviour suddenly jumps into the chaotic mode. In this case chaos comes 'out of the blue sky'. Interestingly though, when looking at the series for r slightly below the critical value 3.82, we can already see the period-3 solution lurking behind: the systems appears to be settling into the period-3 solution, but since it is not yet stable, it escapes from it at seemingly arbitrary instances. This behaviour is called *intermittent*. This is why the transition to chaos is often called *intermittency transition* to(/from) chaos. Note that for increasing r , a period-doubling route to chaos is followed. Indeed, the period-3 solution bifurcates into a period-6, which bifurcates into a period-12, until for some critical value ($r = \dots$) the solution is chaotic again.

2.5 Quantifying chaos: Lyapunov exponents

If the essential aspect of chaos is the sensitive dependence on initial conditions, would it then be possible to quantify this sensitivity? Could we for example measure it? To this end, let us study two series x_n and y_n that are ruled by the same evolution equation but which are initiated by slightly different initial conditions, i.e. $y_0 = x_0 + \epsilon_0$; see the maple code below.

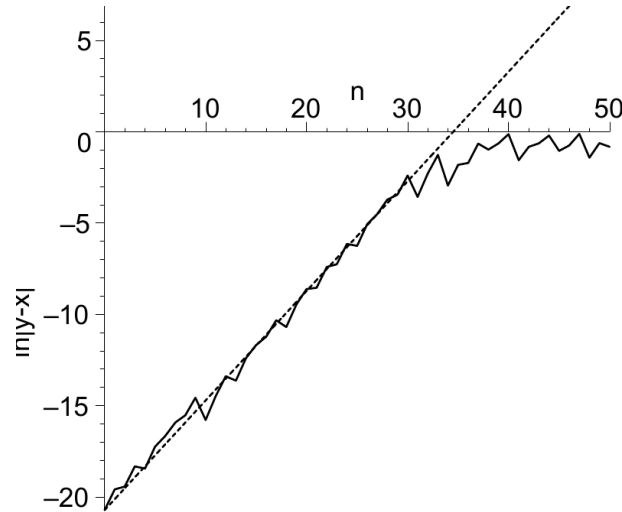


Figure 2.14: Evolution of the deviations $|y_n - x_n|$ plotted on a log-scale.



Maple

```
restart; with(plots): Digits := 40;
f := x -> r * x * (1 - x);
r := 3.9;
N := 100;
X := Array(0..N): Y := Array(0..N):
X[0] := 0.1: Y[0] := X[0] + epsilon;
epsilon := 1e-9;
for n from 0 to N-1 do
    X[n+1] := evalf( f(X[n]) );
    Y[n+1] := evalf( f(Y[n]) );
end do;
```

The two resulting series are plotted in Figure 2.13. It is interesting to see that the difference between the two sets becomes clear already around $n = 25$. Furthermore one gets the impression that the sets diverge quite suddenly. This however is rather deceiving as becomes clear when we plot the logarithm of the absolute differences, $\ln |y_n - x_n|$, see Figure 2.14. The graph shows clearly that the deviations were growing steadily from the very beginning, and that the log-values were merely following a straight line. From this one can conclude that the error growth is exponential, i.e.

$$|y_n - x_n| \simeq |\epsilon_0| e^{\Lambda n} \quad (2.23)$$

The exponent Λ in (2.23) is called the *Lyapunov-exponent*. The exponential growth of deviations turns out to be a generic feature of chaotic systems and by means of Λ one can quantify the rate of growth. The Lyapunov-exponent thus provides a measure of how sensitive the system depends on the initial conditions. In the present example we find $\Lambda \approx 0.6$, based on the slope of the dashed line in Figure 2.14.

To arrive at a more formal definition of Λ , let us follow the evolution of the deviations between the series x_n and y_n , i.e. $\epsilon_n = y_n - x_n$. We consider the situation where the deviations are still so small that a first order Taylor approximation is applicable. In the first step the difference between

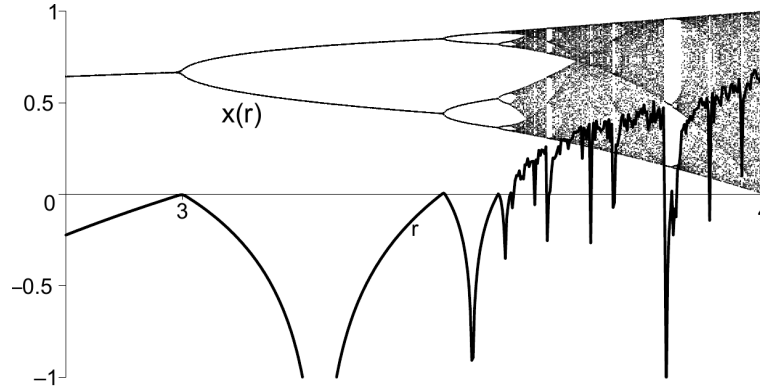


Figure 2.15: Bifurcation diagram $x(r)$ combined with Lyapunov exponents $\Lambda(r)$

$y_1 = f(y_0)$ and $x_1 = f(x_0)$ to first order becomes

$$\epsilon_1 = y_1 - x_1 = f(x_0 + \epsilon_0) - f(x_0) = \epsilon_0 f'(x_0)$$

In the second step we use that $y_1 = x_1 + \epsilon_1$

$$\epsilon_2 = y_2 - x_2 = f(x_1 + \epsilon_1) - f(x_1) = \epsilon_1 f'(x_1) = \epsilon_0 f'(x_0) f'(x_1)$$

Repeating these steps N times, we arrive at

$$\epsilon_N = \epsilon_0 f'(x_0) f'(x_1) f'(x_2) \dots f'(x_{N-1}) \quad (2.24)$$

provided that all ϵ_n are still small enough to justify a Taylor expansion. The Lyapunov exponent introduced in (2.23) can now be determined through $\Lambda = \frac{1}{N} \ln(|\epsilon_N/\epsilon_0|)$, which, by virtue of (2.24), amounts to

$$\begin{aligned} \Lambda &= \frac{1}{N} \ln (|f'(x_0)| |f'(x_1)| |f'(x_2)| \dots |f'(x_{N-1})|) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \ln (|f'(x_n)|) \end{aligned}$$

This result implies that the rate of error-growth can be calculated *a priori* from the evolution of a single series $x_1, \dots, x_N$ and the derivative of the mapping function. As a check we apply (2.25) to the first 25 values of the series x_n plotted in Figure 2.13, from which we obtain $\Lambda = 0.61 \dots$, in good agreement with the slope of the dashed line in Figure 2.14 with an estimated value of 0.6.

From a conceptual point of view, the Lyapunov exponent Λ is very instructive. Λ smaller than zero implies that the deviations become smaller as n increases; consequently the two series converge to each other. Conversely, Λ larger than zero implies that the two series diverge from each other. Again we stress that the rate of divergence is exponential.

This brings us to an important issue: how can one distinguish a chaotic set from a periodic solution with a very large order? The answer is that the chaotic series will have a positive Lyapunov exponent, whereas the periodic series will have a negative Lyapunov exponent. In this context it is illuminating to combine the bifurcation-diagram with a plot of the Lyapunov-exponent as a function of r . The result is shown in Figure 2.15.

The combined plot nicely reveals the relation between the character of the solutions for different r and the corresponding Lyapunov exponents. For $r < 3$ the Lyapunov exponent is always smaller

than zero consistent with the stationary behaviour of the solutions. At $r = 3$, that is at the first bifurcation, $\Lambda = 0$. In fact one sees that at every bifurcation $\Lambda = 0$. Between two consecutive bifurcations the Lyapunov exponent is negative, reaching about halfway very negative values. At these points the solutions are superstable, and the corresponding Lyapunov exponent is minus infinity. The lack of sufficient resolution in r used when plotting Figure 2.15 prohibits Λ from becoming really that small. Note that a value of $\Lambda = -\infty$ implies a convergence rate that is even faster than exponential.

Another crucial point is where Λ becomes positive the first time. This is at $r = r_c = 3.56\dots$, where the subsequent period-doublings have culminated into a solution that can actually be referred to as chaotic. For $r > r_c$ the Lyapunov exponents are predominantly positive, in line with the well filled bands in the bifurcation diagram. Interestingly one can observe within the chaotic range small windows with stable periodic solutions, which comply well with the negative value Lyapunov exponents.

For stable periodic solutions there is direct connection between the (negative) Lyapunov exponent Λ and the stability σ of the period solution. For example, in case of a stable fixed point all $x_n = x^*$ for n sufficiently large. So, provided we disregard the transient phase, equation (2.25) becomes

$$\Lambda = \ln(|f'(x^*)|) = \ln|\sigma|$$

see also (2.10). For a period-2 we have $x_n = x_1^*, x_{n+1} = x_2^*$ in a repetitive manner, hence equation (2.25) reduces to

$$\Lambda = \frac{1}{2} [\ln(|f'(x_1^*)|) + \ln(|f'(x_2^*)|)] = \frac{1}{2} \ln|\sigma^{(2)}|$$

where $\sigma^{(2)}$ is the stability of the period-2 solution as derived in (2.17). The generalisation to period- m is straightforward

$$\Lambda = \frac{1}{m} \sum_{n=1}^m \ln(|f'(x_n^*)|) = \frac{1}{m} \ln|\sigma^{(m)}| \quad (2.25)$$

see also (2.21).

Since the condition for stability of any periodic solution requires $|\sigma^{(m)}| < 1$, equation {eq}\mathtt{eq:disc1d:lyap_period-m} immediately implies that $\Lambda < 0$, consistent with the interpretation of the Lyapunov exponent given above. In addition we can understand why $\Lambda = 0$ exactly at bifurcations, since these points are characterized by $|\sigma^{(m)}| = 1$, i.e. period- m loses its stability, while the period- $2m$ comes into existence. (2.25) also clarifies why in the superstable situation, i.e. when $\sigma^{(m)} = 0$, Λ tends to $-\infty$. The interpretation is that the convergence rate is faster than exponential, in accord with (2.11).

2.5.1 Lyapunov exponents and prediction horizon

Knowledge of the value of the Lyapunov exponent for a certain process, together with the exponential relation (2.23), provides us with the rate at which the uncertainties in the initial condition propagate during subsequent iterations. Generally one does not mind small inaccuracies in the computations, provided that the error in the calculation should stay well below the typical values belonging to the problem under study. For example, in the logistic map the values of x_n range between 0 and 1. Two sets iterated from slightly different initial conditions can be visibly distinguished from each other when their deviations exceed say 1%. So the question is how many steps

N can we iterate the map, assuming an initial inaccuracy of say $\epsilon_0 = 10^{-9}$, before ϵ_N exceeds 1%? Rewriting (2.23) gives the answer

$$N = \frac{1}{\Lambda} \ln |\epsilon_N / \epsilon_0| \quad (2.26)$$

In the example of Figure 2.13 and Figure 2.14, with $r = 3.9$ and $\epsilon_0 = 10^{-9}$, we found $\Lambda = 0.6$, which gives $N \approx 27$. This then is the *prediction horizon*. Proceeding beyond the prediction horizon, in this case proceeding for $N > 27$, makes limited sense, since the inaccuracies are of the same order as the predicted values themselves.

What can we do to move the prediction horizon forward? Since Λ is a characteristic of the process, one cannot change it. So the only option is to decrease the error or uncertainty in the initial condition. But equation (2.26) conveys rather bad news owing to the logarithmic dependence: one wishes for instance to remain accurate for 400 steps, the inaccuracy in the initial condition should be smaller than 10^{-100} , – a costly effort.

But even if it were possible to know the initial conditions with extreme accuracy, then the accuracy of our computer would spoil matters. This process can be best illustrated by the following striking example obtained from [Broer and Verhulst, 1992]. The idea is to iterate two series with exactly the same initial condition. In principle this will yield two identical series, since, although the computer may be inaccurate, it will at least be reproducibly inaccurate. The essence of the example is, however, that the mapping is implemented in two different ways: $x_{n+1} = rx_n(1 - x_n)$ and $y_{n+1} = ry_n - ry_n^2$. See the `maple`-code below:



Maple

```
N := 50; r := 3.9;
X := Array(0..N): Y := Array(0..N):
Y[0] := X[0]; X[0] := 0.1:
for n from 0 to N-1 do
  X[n+1] := evalf( r*X[n]*(1-X[n]) );
  Y[n+1] := evalf( r*Y[n]-r*Y[n]^2 );
  printf("%2d %12.10f %12.10f\n",n,X[n],Y[n]);
end do;
```

Mathematically speaking there should be no difference between the implementations, because $rx(1 - x) = rx - rx^2$. But look at the output

n	X[n]	Y[n]	n	X[n]	Y[n]
0	0.1000000000	0.1000000000	25	0.9242090679	0.9242333454
1	0.3510000000	0.3510000000	26	0.2731820002	0.2731016680
2	0.8884161000	0.8884161000	27	0.7743590205	0.7742168729
3	0.3866184397	0.3866184400	28	0.6814357987	0.6817399150
4	0.9248640251	0.9248640254	29	0.8466160987	0.8461853520
5	0.2710131847	0.2710131840	30	0.5064433925	0.5076072380
6	0.7705036496	0.7705036490	31	0.9748380826	0.9747743060
7	0.6896283246	0.6896283260	32	0.0956623017	0.0958984970
8	0.8347602842	0.8347602820	33	0.3373930004	0.3381377036
9	0.5379486532	0.5379486590	34	0.8718799586	0.8728223283
10	0.9693836087	0.9693836070	35	0.4356506560	0.4329136940
11	0.1157482087	0.1157482150	36	0.9588507313	0.9574477676

(continues on next page)

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12	0.3991671874	0.3991672063	37	0.1538784249	0.1588920060
13	0.9353477013	0.9353477165	38	0.5077794355	0.5212168121
14	0.2358422780	0.2358422260	39	0.9747639733	0.9732444030
15	0.7028607218	0.7028606147	40	0.0959367617	0.1015549670
16	0.8145053574	0.8145055270	41	0.3382583078	0.3558420672
17	0.5892368827	0.5892364660	42	0.8729745372	0.8939521125
18	0.9439434374	0.9439437270	43	0.4324709789	0.3697267590
19	0.2063654751	0.2063644720	44	0.9572153423	0.9088126424
20	0.6387371866	0.6387348891	45	0.1597211100	0.3232016710
21	0.8999327730	0.8999352580	46	0.5234200804	0.8530951686
22	0.3512097306	0.3512019780	47	0.9728608496	0.4887628280
23	0.8886596771	0.8886506793	48	0.1029702060	0.9745075312
24	0.3858802561	0.3859075330	49	0.3602326364	0.0968861510

In this output one can see how the inaccuracies enter at the least significant digit, but steadily move to the more significant digits. At step $n = 45$ the difference between the two series is as large as the values themselves. This teaches us an important lesson: round-off errors, which are always bound to happen, propagate in the same fashion as the uncertainty in the initial conditions does, i.e. with $\sim \exp(\Lambda n)$, where Λ is the Lyapunov exponent of the system. Thus there is no point in knowing the initial conditions with a better accuracy than the accuracy with which your computer performs the calculations.

But if the particular implementation as well as the particular computer does exert an influence, then why, one may wonder, would one plot the series for $N > 50$, as we did in for example in [Figure 2.1](#)? Indeed, it is pointless to the extent that no one can reproduce this plot in a *quantitative manner* if one uses a different computer and/or a different implementation of the mapping. However, in this situation one does obtain a plot that resembles [Figure 2.1](#) in a *qualitative manner*, that is, the plots would be similar in a statistical sense. We will get back to this issue in [section 2.7](#). First we will diagnose in more detail the process of error propagation in a numerical calculation.

2.5.2 Shift map

The Bernoulli shift-map is an interesting map because it reveals in a simple way how inaccuracies in the initial conditions together with computer round-off errors propagate during the iterations. The shift-map is defined by

$$y_{n+1} = 2y_n \bmod 1 \quad (2.27)$$

The chaotic series resulting from (2.27) with $y_0 = 1/\pi$ is shown in [Figure 2.16](#), together with the return-plot. See the maple-code below. Note that we first define the modulo-function `modf`, since it is not a standard function of `maple`.

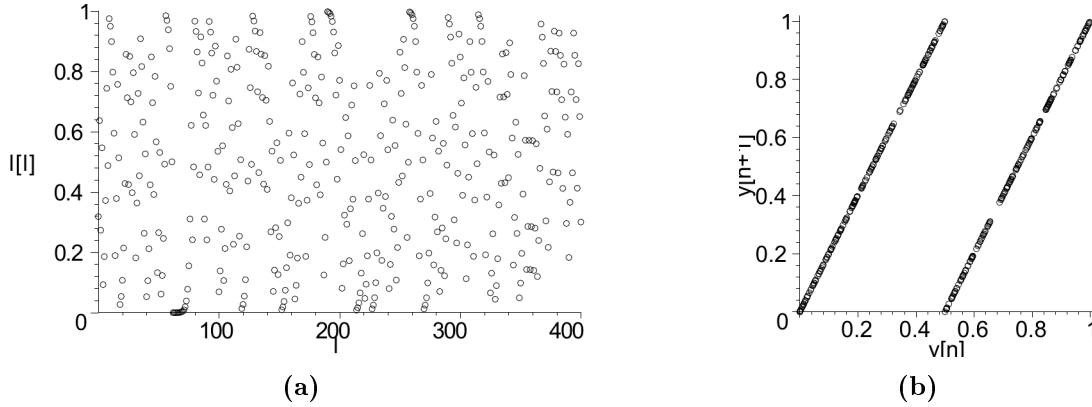


Figure 2.16: Series and return-plot of the shift-map (2.27) (a) Series. (b) Return-plot.



Maple

```
restart; with(plots):
modf := x->x-floor(x);
f := x->modf(2*x);
N :=400;
Y := Array(0..N):
Y[0] := evalf(1/Pi);
for n from 0 to N-1 do
    Y[n+1] := f(Y[n]);
od:
l := seq([n,Y[n]],n=0..N):
pointplot(l,x=0..N,0..1,labels=["n","y[n]"]);
ret := seq([Y[i],Y[i+1]],i=0..N-1):
pointplot(ret,labels=["y[n]","y[n+1]"]);
```

We will make a close study of the evolution of the series $y_0, y_1, \dots$ by looking at the *binary* representation of the numbers. Since the numbers y_n are confined to the range $[0, 1]$, they can be represented by a bit-series $\{b\}$ via: $y = b_1 2^{-1} + b_2 2^{-2} + b_3 2^{-3} + \dots$, where the 'bits' b_i are either 0 or 1. For example, $y = 1/4$ is represented by $\{b\} = \{0100 \dots 0\}$ and $y = 5/8$ by $\{b\} = \{1010 \dots 0\}$. An irrational number like $y = 1/\pi$ or $y = \sqrt{2} - 1$ cannot be represented by a finite bit-series; the computer representation of such a number will therefore be inaccurate.

Table 2.1: Propagation of inaccuracies in the shift-map visualized as a flow in the binary representation of the numbers.

	2^{-1}	2^{-2}	2^{-3}	2^{-4}	$\dots$	2^{-K+3}	2^{-K+2}	2^{-K+1}	2^{-K}
y_0	0	1	0	1	$\dots$	1	1	0	?
y_1	1	0	1		$\dots$	1	0	?	?
y_3	0	1			$\dots$	0	?	?	?
y_4	1				$\dots$	?	?	?	?
$\vdots$									
y_{K-3}	1	1	0	?	?	?	?	?	?
y_{K-2}	1	0	?	?	?	?	?	?	?
y_{K-1}	0	?	?	?	?	?	?	?	?
y_K	?	?	?	?	?	?	?	?	?

Suppose that we know the initial condition y_0 accurately up to bit number K . Whether the limited accuracy is due to lack of precise knowledge of the initial condition, or due to the limited computer precision, is irrelevant for the example. In either case it means that the bits below K are specified and those beyond K are unknown. See Table 2.1 where we have indicated the unknown bits by a question mark. What can we say about y_1 ? Applying (2.27) involves a multiplication by 2, which, in the bit-series representation, amounts to shifting all bits one position to the left. The modulo-function makes sure that the numbers will stay within $[0, 1]$, meaning that the left-most bit is simply shifted out of the domain, and can henceforth be disregarded. The relevant point to notice in Table 2.1 is that during the multiplication operation the unknown bit shifts one position to the left, while a new unknown bit enters from the right – a process that continues for each subsequent multiplication. We observe that already after K steps even the most-significant bit is no longer known: the entire bit-series is vacated by question marks; they can be either 0 or 1, but we have no way of knowing. Clearly we have come beyond the prediction horizon, and no longer can we identify a relation between y_K and the initial value y_0 . The Lyapunov exponent of this map can be found by realizing that each step the error grows by a factor 2, i.e. $\epsilon_n/\epsilon_0 \sim 2^n$. The Lyapunov exponent of the shift-map is therefore $\Lambda = \ln 2$.

It might seem that the error propagation as illustrated in Table 2.1 only holds for the shift-map, but below we will show that this underlying principle is more general. Consider the following transform of variables

$$x_n = \frac{1 - \cos 2\pi y_n}{2} \quad (2.28)$$

Next we work out x_{n+1} and try to express it in terms of x_n :

$$\begin{aligned} x_{n+1} &= \frac{1 - \cos 2\pi y_{n+1}}{2} = \frac{1 - \cos(4\pi y_n \bmod 2\pi)}{2} = \frac{1 - \cos(4\pi y_n)}{2} \\ &= 1 - \cos^2(2\pi y_n) = (1 - \cos 2\pi y_n)(1 + \cos 2\pi y_n) = 4x_n(1 - x_n) \end{aligned}$$

An unexpected and surprising result: the logistic map (2.4) for $r = 4$ is intimately related to the shift-map. In effect, there is no real difference between iterating the logistic map, or iterating the shift-map and subsequently applying (2.28). It shows that the Lyapunov-exponent of the logistic map for $r = 4$ is $\Lambda = \ln 2$.

Finally we mention one more related mapping, the so-called tent-map

$$z_{n+1} = 1 - 2 \left| z_n - \frac{1}{2} \right| \quad (2.29)$$

Series and returnplot in Figure 2.17.

Similar to the shift-map, a relation can be established with the $r = 4$ logistic map. The transformation is

$$x_n = \frac{1 - \cos \pi z_n}{2} \quad (2.30)$$

2.6 Analytical solutions to chaotic series

The title of this section may sound rather paradoxical – after all, if the point of chaos is the impossibility to *compute* the system over a sufficiently long period of time, then certainly we would not be able to write down the solution in a closed form, would we? Surprisingly, however,

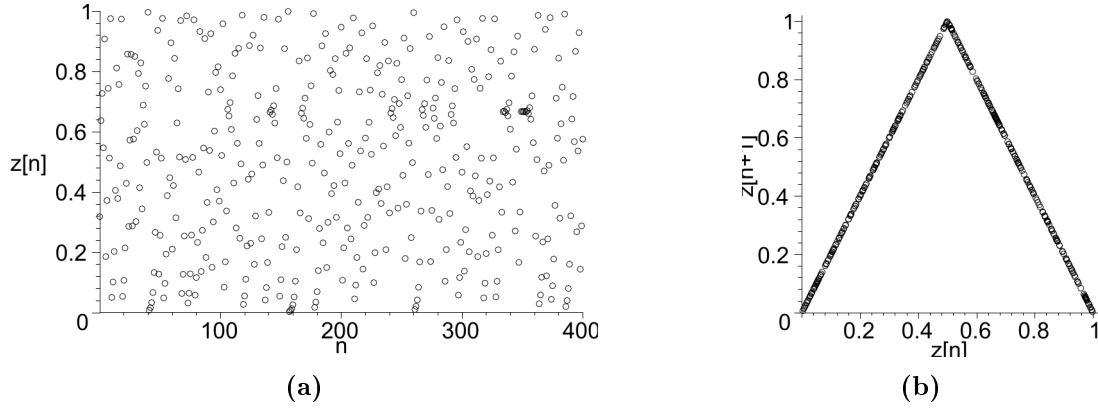


Figure 2.17: Series and return-plot of the tent-map (2.29) (a) Series. (b) Return-plot.

it turns out that occasionally one can find closed form solutions in the chaotic range. For example, the formula below provides the solution to the logistic map (2.4) at $r = 4$

$$x_n = \sin^2(2^n \theta) \quad \text{with } \theta \text{ such that } x_0 = \sin^2(\theta)$$

Indeed:

$$\begin{aligned} x_{n+1} &= \sin^2(2^{n+1}\theta) = \sin^2(2 \cdot 2^n \theta) = [2 \sin(2^n \theta) \cos(2^n \theta)]^2 \\ &= 4 \sin^2(2^n \theta) \cos^2(2^n \theta) = 4 \sin^2(2^n \theta) [1 - \sin^2(2^n \theta)] \\ &= 4x_n(1 - x_n) \end{aligned}$$

Before we interpret this fascinating result, let's focus on a method to formally derive a closed form solution to a map like

$$x_{n+1} = f(x_n) \tag{2.31}$$

Assume the solution x_n can be expressed as

$$x_n = F[a^n \theta] \quad \text{with } \theta \text{ such that } x_0 = F[\theta]$$

then the function F would have to satisfy

$$F[az] = f(F[z]) \tag{2.32}$$

To see this take $z = a^n \theta$, then the left-hand side and right-hand side of (2.31) are respectively

$$\begin{aligned} x_{n+1} &= F[a^{n+1}\theta] = F[az] \\ f(x_n) &= f(F[a^n \theta]) = f(F[z]) \end{aligned}$$

So one has obtained a solution to (2.31) if one is able to find a function F that satisfies (2.32). This is however not so easy for general mappings f . So usually it works the other way around: if one knows a non-trivial relation for (2.32), one can derive the corresponding mapping function f for which one has just found a closed-form solution. Below we work out a few examples which follow from doubling formulas, like $\sin(2z)$ or $\tanh(2z)$ or a tripling formula like $\sin(3z)$.

$$\begin{aligned} \sin(2z) &= 2 \sin(z) \cos(z) \rightarrow \sin(2z)^2 = 4 \sin^2(z)(1 - \sin^2(z))^2 \\ F[z] &= \sin^2(z) \rightarrow F[2z] = 4F[z](1 - F[z]) \\ &\rightarrow f(x) = 4x(1 - x) \end{aligned}$$

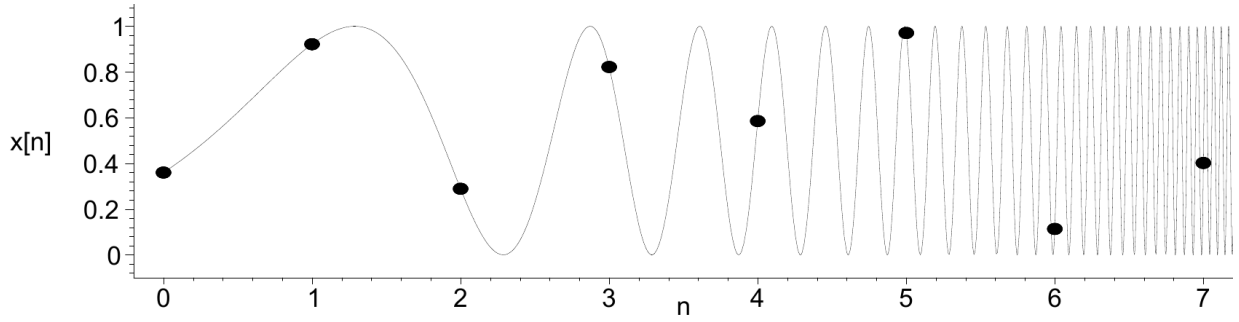


Figure 2.18: The analytical solution $x_n = \sin^2(2^n \theta)$ from (2.33) for $x_0 = 0.36$ ($\theta = 0.64\dots$) together with the first few iterations $x_1, \dots, x_7$ of the logistic map for $r = 4$.

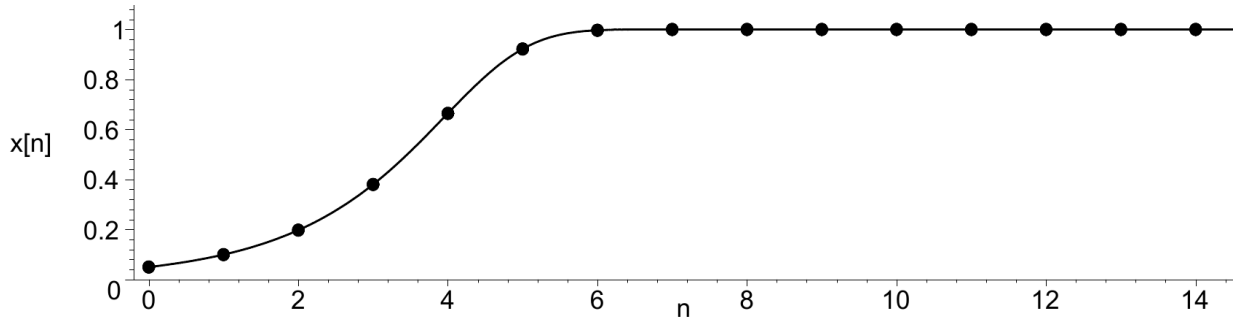


Figure 2.19: The analytical solution $x_n = \tanh(2^n \theta)$ from (2.34) for $x_0 = 0.05$ ($\theta = 0.05\dots$) together with the first few iterations $x_1, \dots, x_{14}$ of the map $x_{n+1} = 2x_n / (1 + x_n^2)$.

So the corresponding map and closed form solution are

$$x_{n+1} = 4rx_n(1 - x_n) \quad x_n = \sin^2(2^n \theta) \quad (2.33)$$

In Figure 2.18 we have plotted the analytical solution together with the first few iterations obtained from the mapping.

Next we work out the doubling formula for $\tanh(z)$

$$\begin{aligned} \tanh(2z) &= \frac{2 \tanh(z)}{1 + \tanh^2(z)} & F[z] = \tanh(z) &\rightarrow F[2z] = \frac{2F[z]}{1 + F^2[z]} \\ & & &\rightarrow f(x) = \frac{2x}{1 + x^2} \end{aligned}$$

The corresponding map and closed form solution are therefore

$$x_{n+1} = \frac{2x_n}{1 + x_n^2} \quad x_n = \tanh(2^n \theta) \quad (2.34)$$

The results are plotted in Figure 2.19. Apparently there is no chaotic behaviour and the system settles into the fixed-point $x^* = 1$. Note also that the system is superstable, as $f'(x^*) = 0$.

In our last example we elaborate the tripling formula for $\sin(z)$, which yields

$$\begin{aligned} \sin(3z) &= -4\sin^3(z) + 3\sin(z) & F[z] = \sin(z) &\rightarrow F[3z] = -4F^3[z] + 3F[z] \\ & & &\rightarrow f(x) = -4x^3 + 3x \end{aligned}$$

Hence the corresponding map and closed form solution are

$$x_{n+1} = -4x_n^3 + 3x_n \quad x_n = \sin(3^n \theta) \quad (2.35)$$

While these solutions are definitely entertaining, they bring us no closer to an accurate prediction of x_n for large n . Although it seems that the closed form solutions are now computable for arbitrary n , they still possess at their heart the key problem of chaos: sensitivity on initial conditions. Take for example expression (2.33). An error in the initial condition translates directly to an error in θ , say ϵ_0 . The solution

$$x_n = \sin^2(2^n[\theta + \epsilon_0]) = \sin^2(2^n \theta + 2^n \epsilon_0)$$

The error in the argument grows as $2^n \epsilon_0$ which may be small compared to $2^n \theta$ but that notion is irrelevant because the sine function works modulo 2π , which implies that the effect of $2^n \epsilon_0$ comes into play rather soon.

The closed-form solutions are also useful to the extent that they provide us direct insight into the error-growth and therefore the value of the Lyapunov-exponent. When $2^n \epsilon_0$ is still small the deviation of two sets with slightly perturbed initial conditions evolves as

$$|\sin^2(2^n[\theta + \epsilon_0]) - \sin^2(2^n \theta)| = |\sin^2(2^n \theta + 2^n \epsilon_0) - \sin^2(2^n \theta)| \simeq |\sin(2^{n+1} \theta) \epsilon_0| 2^n$$

This shows that $\delta_n \sim 2^n = \exp(\Lambda n)$, with $\Lambda = \ln 2$, consistent with the finding mentioned in section 2.5.2. In the same vein we conclude that $\Lambda = \ln 3$ for the mapping of (2.35).

Finally it is instructive to determine the Lyapunov exponent of (2.34); the deviations evolve according to

$$|\tanh(2^n[\theta + \epsilon_0]) - \tanh(2^n \theta)| = |(1 - \tanh^2(2^n \theta)) \epsilon_0| 2^n$$

It would seem that $\delta_n \sim 2^n$, i.e. $\Lambda = \ln 2$, but closer inspection tells us that the first term approaches zero rapidly. Working out the asymptotics of $1 - \tanh^2(y) \sim 4 \exp(-2y)$, yields

$$|1 - \tanh^2(2^n \theta)| 2^n \sim 4 \exp(-2^{n+1} \theta) 2^n = 4 \exp(-2^{n+1} \theta + n \ln 2) \rightarrow 0$$

i.e. $\Lambda \rightarrow \infty$, consistent with the earlier remark above that the system is superstable.

2.7 A statistical approach to deterministic chaos

The message of sections 2.5 and 2.6 is that in case of chaotic behaviour there is no use in following an analytical or computational approach in predicting the (far) future state of the system. One might have the analytical solution e.g. (2.33) or a straightforward computational scheme like $x_{n+1} = f(x_n)$ to calculate x_n , but in the end accumulating errors render both approaches useless. The system behaves in a seemingly random fashion. In fact it does make sense to treat the *deterministic* chaotic system as a *stochastic* system. Formally the probability density function (pdf) is given by

$$p(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta(x - x_n) \quad (2.36)$$

with δ the Dirac delta-function. There is a variety of methods to elaborate (2.36)

, but they are beyond our scope. Here we show how one can find the pdf in some particular cases.

The pdf of the logistic map for $r = 4$ can for example be found by employing its relation to the shift-map (2.28). Using the transformation rule for densities

$$p(x)dx = p(y)dy \quad (2.37)$$

and the fact that the shift-map data y_n are uniformly distributed, $p(y) = 1$ between 0 and 1, giving

$$p(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad x \in (0, 1) \quad (2.38)$$

Note that the same result could be derived from the analytical solution (2.33):

$$x_n = \sin^2(2^n\theta)$$

Since $2^n\theta \bmod 2\pi$ will visit values between 0 and 2π uniformly, we can elaborate (2.36) into

$$\begin{aligned} p(x) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta(x - x_n) = \frac{1}{2\pi} \int_0^{2\pi} \delta(x - \sin^2(t)) dt \\ &= \frac{2}{\pi} \int_0^{\pi/2} \delta(x - \sin^2(t)) dt \end{aligned}$$

Substitution of $y = \sin^2 t$, with $dy = 2 \sin(t) \cos(t) dt = 2\sqrt{y}\sqrt{1-y} dt$ then leads to the same result as (2.38).

$$p(x) = \frac{1}{\pi} \int_0^1 \frac{\delta(x-y)}{\sqrt{y(1-y)}} dy = \frac{1}{\pi\sqrt{x(1-x)}}$$



Example

The `maple` code below determines the pdf based on $N = 8192$ iterations of the logistic map and plots the result together with the expression derived in (2.38), producing Figure 2.20.

```
restart; with(plots): with(stats[statplots]):
f := x -> 4 * x * (1 - x);
N := 8192; X := Array(0..N):
X[0] := 0.1:
for n from 0 to N-1 do X[n+1] := evalf( f(X[n]) ); end do:
pdf := x -> 1/(Pi*sqrt(x*(1-x)));
p1 := histogram(convert(X[0..N], list), numbars=32,color=gray):
p2 := plot(pdf(x),x=0.025..0.975,thickness=5,color=black):
display([p1,p2],view=[0..1,0..2],labels=["x","p(x)"]);
```

In a similar way we can find the pdf for mapping (2.35) for which we know that

$$x_n = \sin(3^n\theta)$$

$$\begin{aligned} p(x) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta(x - x_n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \delta(x - \sin(t)) dt \\ &= \frac{1}{\pi} \int_{-1}^1 \frac{\delta(x-y)}{\sqrt{1-y^2}} dy = \frac{1}{\pi\sqrt{1-x^2}} \end{aligned}$$

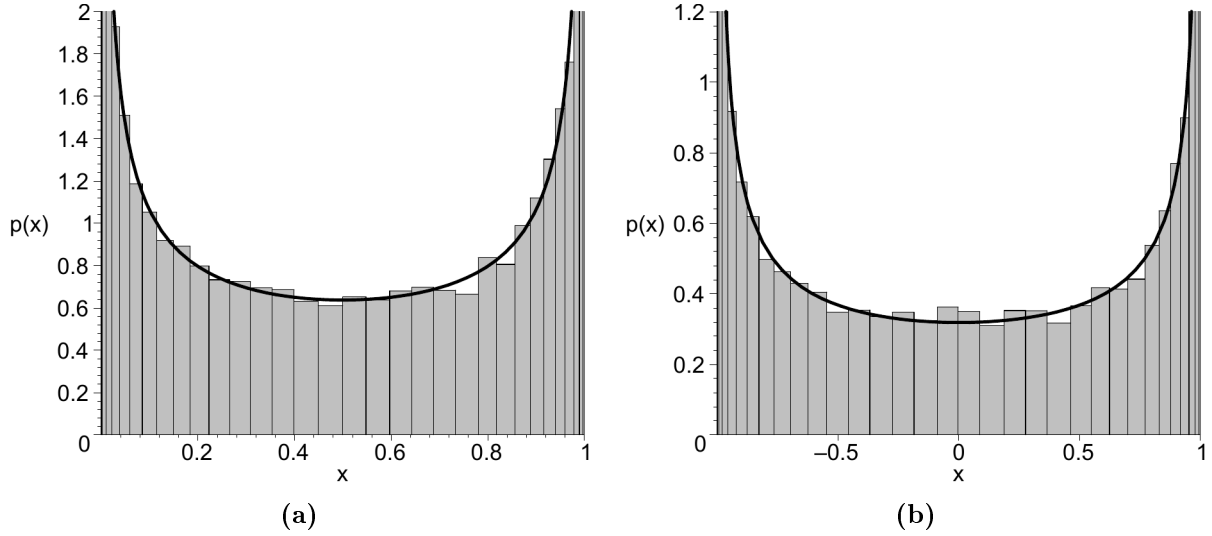


Figure 2.20: pdfs (a) Pdf for the logistic map at $r = 4$; solid line represents (2.38). (b) Pdf for the mapping $x_{n+1} = -4x_n^3 + 3x_n$; solid line represents (2.39).

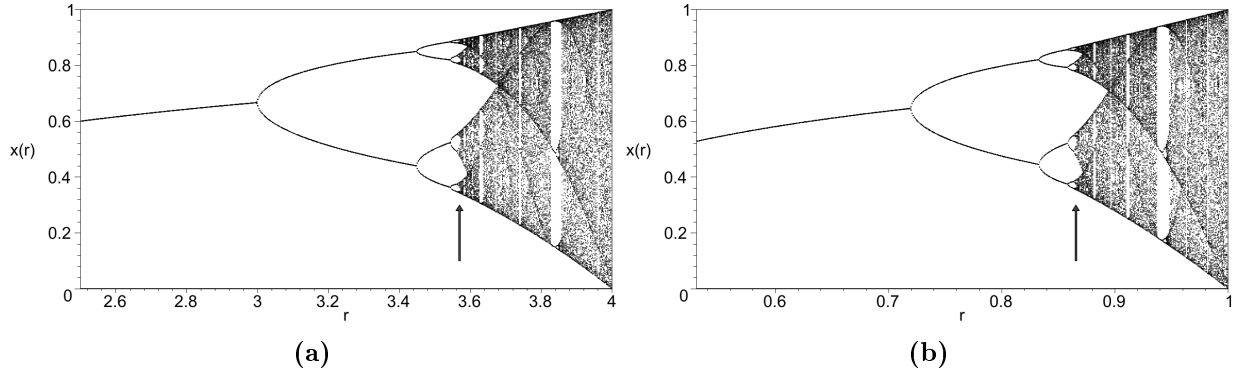


Figure 2.21: Period-doubling in two different mappings. (a) Period doubling in the logistic map until $r_\infty = 3.56994537$ (arrow). (b) Period doubling in the sine-map until $r_\infty = 0.86557928$ (arrow).

2.8 The universal period-doubling route to chaos

If one compares in Figure 2.21 the bifurcation diagram of the logistic map with that of the sine map, given by respectively,

$$x_{n+1} = rx_n(1 - x_n) \quad r \in [0, 4] \quad x_{n+1} = r \sin(\pi x_n) \quad r \in [0, 1]$$

one might be struck by the qualitative similarity of the graphs. Both exhibit a period doubling route to chaos with the subsequent bifurcations occurring progressively 'faster'. More precisely, if we denote by r_p the value of r at the onset of a period 2^p , then it appears that for both iterative maps the distance $r_{p+1} - r_p$ strongly decreases for increasing p . Using the techniques outlined in section 2.3 we can calculate r_p for the logistic map for the first bifurcations $r_1 = 3, r_2 = 1 + \sqrt{6} = 3.449489743, r_3 = 3.544090360$, which indeed shows that the period-doublings progressively densify on the r -axis, until the border to chaos is reached at $r_\infty = 3.56994537$ (indicated by the arrow). For the sine-map the same happens, $r_1 = 0.7199616829, r_2 = 0.8332663536, r_3 = 0.8586090598, \dots$ until $r_\infty = 0.86557928$ is reached. It was Feigenbaum who made the remarkable discovery that this similarity is not only qualitative, but also quantitative. If one measures the

relative distances between successive bifurcations

$$\delta_p = \frac{r_p - r_{p-1}}{r_{p+1} - r_p} \quad (2.39)$$

then, for $p \rightarrow \infty$, δ_p approaches a *universal* number $\delta = 4.6692\dots$, now called the Feigenbaum constant. Feigenbaum proved mathematically that this holds for *any* iterative map of the form $f_r(x) = rq(x)$, where $q(x)$ is a differentiable function with only *one* quadratic maximum; the latter means that a Taylor expansion around the maximum has lowest order 2. Note that both $q(x) = x(1-x)$ and $q(x) = \sin(\pi x)$ satisfy these criteria, but also, say, $q(x) = \exp(-x^2)$.

We will not go into the mathematical details of this complicated proof; for a very nice 'pedestrian' treatment one is referred to [Strogatz, 1994]. Rather, we show a simple way to get a better appreciation of the regularity of the period-doubling. It is not easy to see this in a normal bifurcation diagram because the successive bifurcations densify really fast according to (approximately)

$$r_p \simeq r_\infty - c\delta^{-p} \quad (2.40)$$

which can be verified by substituting (2.40) into (2.39) and assuming that $\delta_p \simeq \delta$. The values of r_∞ and c depend on the specific mapping. (2.40) inspires us to plot the bifurcation diagram not as a function of r , but as a function of a new parameter a :

$$a = -\ln[r_\infty - r] \Rightarrow a_p \simeq p \ln \delta - \ln c \quad (2.41)$$

Plotted as a function of a , bifurcations thus show up in an equidistant fashion with separation equal to $\ln \delta$. The resulting diagrams are plotted in Figure 2.22. The plots, which were generated with a `maple`-code similar to the one shown below, now nicely reveal the regular spacing between the subsequent bifurcations.



Maple

```
restart; with(plots):
f := x -> r * sin(Pi*x);
rc := 0.86557928;
Na := 500; rmin := 0.3;
amin := -log(rc - rmin); amax := 10.2;
N := 1000; Nmin := ceil(0.8*N);
X := Array(0..N): X[0] := 0.57:
for i from 0 to Na do
  a := evalf(amin + (amax - amin)*i/Na);
  r := rc - exp(-a);
  for n from 0 to N-1 do X[n+1] := evalf( f(X[n]) ); end do:
  pl[i] := seq([a,X[n]],n=Nmin..N);
end do:
pointplot([seq(pl[i],i=1..Na)],view=[amin..amax,0.33..0.9],
  labels=["a", "x(a)"],color=black,axes=BOXED);
```

Yet if you would wish to make a similar plot for a different mapping, you find that you need to know *a priori* the value of r_c for that mapping and you might wonder how we derived those values for the two mappings. This is not so easy, but below we explain a way to do it. First it must be noted that in figure Figure 2.22 the big dots do not indicate locations of bifurcations, but rather locations of 'superstability', i.e. where the stability $\sigma = 0$ (see section 2.2). For example a superstable period- m solution x_k^* , ($k = 1, \dots, m$) then satisfies (see also section 2.3.2 for a

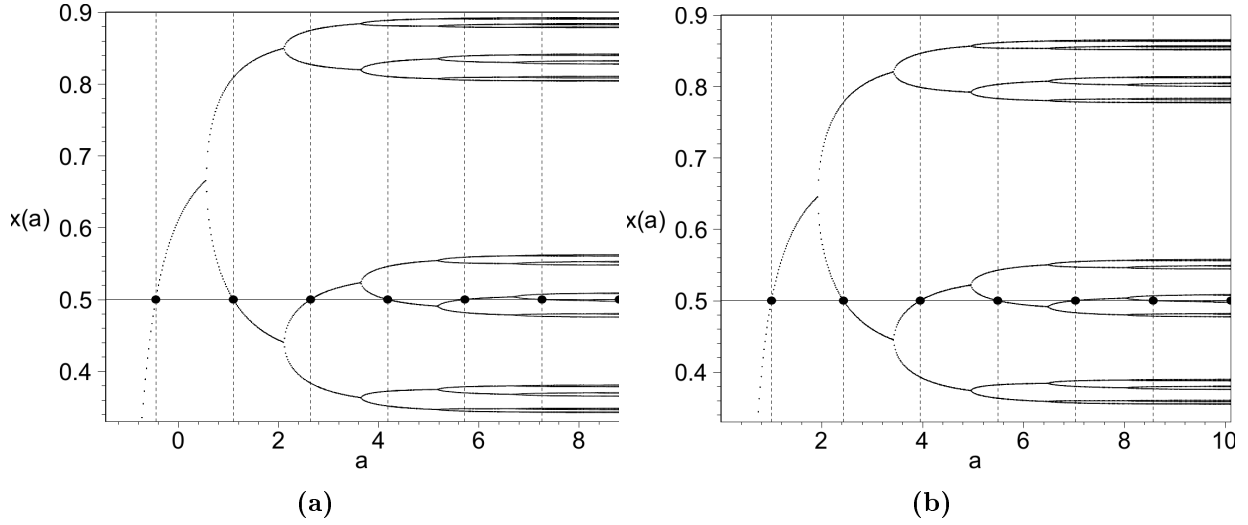


Figure 2.22: Graphs revealing the universal period-doubling for the logistic map and the sine map. (a) Bifurcation diagram of the logistic map $x_{n+1} = rx_n(1 - x_n)$ as a function of $a = -\ln(r_\infty - r)$ with $r_\infty = 3.56994537$. (b) Bifurcation diagram of the sine map $x_{n+1} = r \sin(\pi x_n)$ as a function of $a = -\ln(r_\infty - r)$ with $r_\infty = 0.86557928$.

discussion of stability of periodic solutions)

$$\forall_{k \in \{1, \dots, m\}} : f^{(m)}(x_k^*) = x_k^* \quad \sigma^{(m)} = \prod_{k=1}^m f'(x_k^*) = 0$$

Consequently, mappings of the form $f_r(x) = rq(x)$, as considered by Feigenbaum, are superstable when at least one of the m -periodic points satisfies $q'(x_k^*) = 0$. For the sine-map $q'(x) = \pi \cos \pi x$ and for the logistic map $q'(x) = 1 - 2x$, so in both cases one finds that $q'(1/2) = 0$. Therefore, whenever one of the periodic branches crosses $x = 1/2$ in the corresponding bifurcation diagrams, we know that the solution is superstable.

The advantage of focussing on superstable points is that the convergence is extremely fast compared to points of bifurcation which are associated with very slow convergence, which can also be seen from the corresponding Lyapunov exponents $\Lambda \rightarrow -\infty$ and $\Lambda = 0$, respectively. In principle the foregoing paves the way towards rapidly determining the locations r_p and finding δ_p . Since we know we must look at $x = 1/2$ and find r such that

$$f_r^{(2^p)}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)$$

Execution of the simple maple commands below



Maple

```
f := x -> r*x*(1-x):
fsolve((f@@8)(1/2) = 1/2, r, 2..3.6);
```

directly gives the first four values of r of superstability (recall that a period 8 also solves for periods 1,2,4)

2., 3.236067977, 3.498561699, 3.554640863

Unfortunately this method breaks down for period 16 and higher due to practical reasons; $f^{(m)}(1/2)$ is a polynomial in r with leading order r^{2^m-1} . For $m = 8$ this gives already r^{255} ; for period-16 it amounts to r^{65535} which is hardly manageable of course.

Seed at x_0 where x_0 satisfies $q'(x_0) = 0$ to satisfy the superstability criterion. For the logistic map and the sine map $x_0 = 1/2$. Next we pick some value of r and generate

$$x_1(r) = f(x_0), x_2(r) = f(x_1), \dots, x_m(r) = f(x_{m-1})$$

If r is such that $x_m(r) = x_0$ we have found a superstable period m solution. To find r we use the well-known Newton-Raphson iterative method

$$r_{i+1} = r_i - \frac{g(r_i)}{\frac{dg}{dr}(r_i)} \quad (2.42)$$

with

$$g(r) = x_m(r) - x_0$$

Finding the derivative $dg(r)/dr = dx_m(r)/dr$ needs some extra effort. One can express $x_m(r)$ as $f[x_{m-1}(r)]$, which is equal to $rq[x_{m-1}(r)]$, and use the product rule and the chain rule of differentiation to find a recursive relation:

$$\begin{aligned} \frac{d}{dr}x_m(r) &= \frac{d}{dr}\{f[x_{m-1}(r)]\} = \frac{d}{dr}\{rq[x_{m-1}(r)]\} \\ &= q[x_{m-1}(r)] + r \frac{d}{dr}\{q[x_{m-1}(r)]\} \\ &= \frac{1}{r}f[x_{m-1}(r)] + r \frac{\partial q}{\partial x}[x_{m-1}(r)] \frac{d}{dr}x_{m-1}(r) \\ &= \frac{1}{r}x_m(r) + f'[x_{m-1}(r)] \frac{d}{dr}x_{m-1}(r) \end{aligned}$$

where we also used $r\partial q/\partial x = f'(x)$. Below find the maple implementation. First we define $f(x, r)$, $f'(x, r)$ and $h(r)$, which the right-hand side of (2.42); next we loop over p from zero to 11, thus addressing period-0 to period-2048, and perform the Newton-Raphson iteration until a stop-criterion (related to the numerical precision) has been reached. It turns out that a good initial guess for r_p helps a lot in finding the correct value, in particular for larger p . We therefore calculate the Feigenbaum numbers δ_p of previous periods when possible and use it to guess what the next value of r_p will be. The subsequent Newton-Raphson iteration then refines this value. The **maple** code will give numbers such as given in table Table 2.2; for the sine-map one should replace $r[0]$ by an appropriate number ($r[0] = 0.9$ works) to get the first 3 values of r_p right.



Maple

```

restart: with(plots):
Digits := 20; stopcrit := 10^(-Digits+2):
f := (x,r)-> r*x*(1-x);
df := unapply(diff(f(x,r),x),[x,r]);
# define function for Newton-Raphson procedure
h := proc(r,m)
local n,X,dg,result;
X := Array(0..m); X[0] := 1/2;
for n from 0 to m-1 do X[n+1] := evalf(f(X[n],r)): end do:
dg := 0; for n from 1 to m do dg := evalf(X[n]/r + df(X[n-1],r)*dg); end
do:
result := r - (X[m]-X[0])/dg;
end proc:
P := 11; # calculations until period 2^11 = 2048
for p from 0 to P do
m := 2^p;
# use previous value of delta for a smart start value of r
if( p >= 3 ) then
delta[p-2] := (rc[p-2]-rc[p-3])/(rc[p-1]-rc[p-2]);
r[0] := rc[p-1]+(rc[p-1]-rc[p-2])/delta[p-2];
else r[0] := 3.57;
end if:
# Newton-Raphson iteration to find zero-crossing of h
eps := 1;
for i from 0 to 100 while eps > stopcrit do
r[i+1] := evalf(h(r[i],m));
eps := abs(r[i+1]-r[i]);
end do:
rc[p] := r[i];
end do:
# print results of r and delta
seq( print(p,rc[p],(rc[p]-rc[p-1])/(rc[p+1]-rc[p])),p=1..P-1);

```

The universality of δ in (quadratic unimodal) maps is a truly fascinating element, and $\delta = 4.67$ has been 'measured' in a variety of physical systems. But it may be good to put the latter result in some perspective because one might get the false impression that physical systems should obey the universal behaviour. This is not necessary at all as will become clear when we revisit the assumptions: 1) that the mapping is one-dimensional, 2) that the mapping has only one maximum (unimodal), 3) that this maximum is quadratic. Let us investigate the impact of the latter assumption and look at the following map in more detail:

$$f(x) = r(1 - |2x - 1|^\eta) \quad r \in [0, 1] \quad \eta \geq 1 \quad (2.43)$$

This map has a maximum that is not quadratic, except when $\eta = 2$ for which we recover the logistic map.

Mapping (2.43) gives us a possibility to find the dependence of δ on the value of η by using the procedure outlined above. Adapting the maple code slightly gives the data-points plotted in Figure 2.23, showing values of δ ranging from 3 to as much as 7 for an order-4 maximum $\eta = 4$.

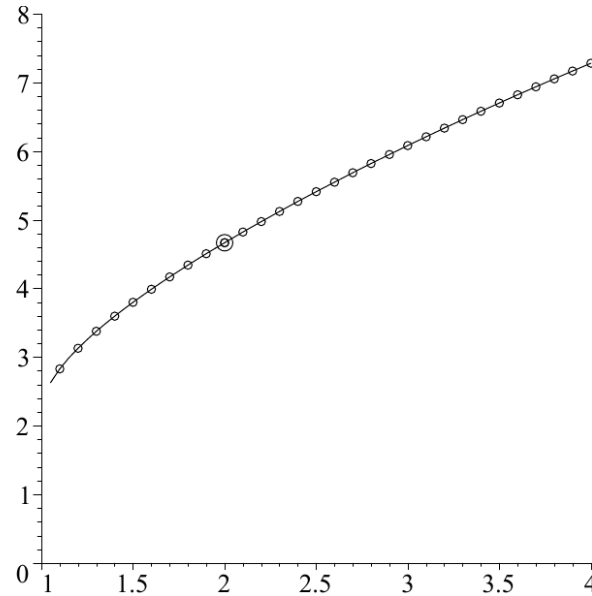


Figure 2.23: Feigenbaum number δ as a function of η for the map $f_\eta(x) = r(1 - |2x - 1|^\eta)$. The mapping f_η represents the logistic map for $\eta = 2$, in which case the 'universal' value of $\delta = 4.669\dots$ is retrieved. For $\eta \rightarrow 1$ the map becomes equal to the tent-map (2.29), which has no period-doubling route. This explains the bend near $\eta \rightarrow 1$.

Table 2.2: $\delta_p = (r_p - r_{p-1})/(r_{p+1} - r_p)$ for the logistic map and the sine-map produced by the maple code. As shown, the values of δ_p rapidly converge for both maps to the Feigenbaum constant $\delta = 4.6692\dots$

LOGISTIC MAP			SINE MAP		
p	r[p]	delta[p]	p	r[p]	delta[p]
0	2.00000000	0	0.50000000		
1	3.23606798	4.708943	1	0.77773377	4.045742
2	3.49856170	4.680771	2	0.84638217	4.555853
3	3.55464086	4.662960	3	0.86145035	4.645182
4	3.56666738	4.668404	4	0.86469418	4.664075
5	3.56924353	4.668954	5	0.86538967	4.668106
6	3.56979529	4.669157	6	0.86553866	4.668967
7	3.56991347	4.669191	7	0.86557057	4.669151
8	3.56993877	4.669199	8	0.86557741	4.669191
9	3.56994419	4.669201	9	0.86557887	4.669199
10	3.56994536	4.669201	10	0.86557918	4.669201

2.9 Exercises

2.9.1 Newton-Raphson method

Downloads: `NewtonRaphson.mw`, `ex0093r00.eps`, `NewtonRaphson.fig`

The Newton-Raphson method provides a fast way to find solutions to equations like $3 \tanh(x) = x^3$. More in general it will yield a zero-crossing x^* of a function $g(x)$, i.e. a value of x for which

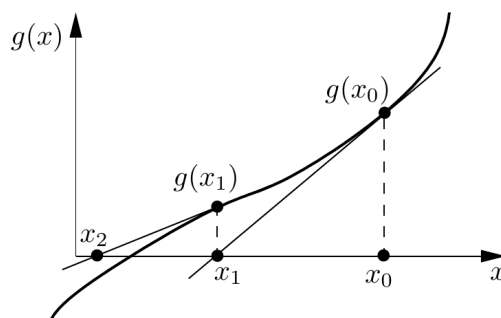


Figure 2.24: A sketch of how the Newton-Raphson method iteratively locates a zero-crossing of $g(x)$.

$g(x^*) = 0$. In the example above one would take $g(x) = 3 \tanh(x) - x^3$. Starting with the initial value x_0 sufficiently close to x^* , the recipe for finding x^* is to iterate the one-dimensional discrete map

$$x_{n+1} = f(x_n) \quad \text{with} \quad f(x) = x - \frac{g(x)}{g'(x)} \quad (2.44)$$

Note that x^* is a fixed point for the map f .

- Take $g(x) = 3 \tanh(x) - x^3$ and program the Newton-Raphson method in **maple** by iterating the map (2.44) until $N = 20$. Start with $x_0 = 1.0$ and show the series.
- Calculate the stability of the fixed point x^* . Does this answer explain the rapid convergence of the method?
- Determine more in general the stability of the fixed point (zero crossing) x^* for *arbitrary* functions g .

We will now study the method for the following function

$$g(x) = \left[\cosh \left(4 - \frac{3}{x} \right) - 1 \right]^{1/a} \quad (2.45)$$

- Take $a = 6$ and use the same program of question (a). Start with $x_0 = 0.4$ and show the series until $N = 100$. Show also the return-plot.
- What is the stability of the fixed point $x^* = 3/4$? How can you match this answer with your answer to question (c)?
- Show a bifurcation diagram for $a = 2 \dots 6$. At which value of a is the fixed point no longer stable?
- Study the bifurcation diagram and look for a value of a for which a period-2 solution occurs. Prove its stability for this particular value of a .

2.9.2 Cubic map

Downloads: CubicMap.mws

Consider the variation on the logistic map:

$$x_{n+1} = \lambda x_n \left(1 - \frac{x_n^2}{\lambda} \right) \quad 0 < \lambda < 3$$

λ is the free parameter of the mapping; we will take here $0 < \lambda < 3$. $\{x_n\}$ is real valued.

- a) Determine the fixed points. Discuss the role of λ on the number of fixed points. Determine the stability of the fixed points (except for $\lambda = 1$).
- b) Determine the maximum value of the initial condition x_0 for which the series $\{x_n\}$ stays finite for all n (hint: use the cobweb diagram).
- c) $\lambda = 1$ is a special case. Use the cobweb diagram to discuss the stability of the fixed point(s) for the case $\lambda = 1$. Calculate the region of attraction of the fixed point(s).
- d) Consider the case $\lambda = 1.5$. Discuss the regions of attraction of the stable fixed point(s).
- e) Consider the case $\lambda = 1.5$. Is a period 2 solution of the mapping possible? If yes, find the initial conditions for which the period 2 solution is found. Discuss the stability of the trajectory. If no, provide a proof that this is not possible.
- f) Consider the case $\lambda = 2.2$. Is a period 2 solution of the mapping possible? Discuss the stability of the period 2 solution.
- g) Discuss the possibility of a period 3 solution, $x_n \rightarrow x_{n+1} \rightarrow x_{n+2} \rightarrow x_{n+3} = x_n$, with $x_n \neq x_{n+1}$ and $x_n \neq x_{n+2}$ for $\lambda = 2.5$.

2.9.3 Universality revisited

Downloads: `UniversalityCounterExamples.mws`

- a) Make bifurcation diagrams of the following map:

$$x_{n+1} = r(1 - |2x - 1|^\eta) \quad x_0 \in [0, 1] \quad r \in [0, 1]$$

for $\eta = 2$, and $\eta = 0.75$. For which values of η do you observe the universal period doubling route, i.e. obeying Feigenbaum's constant? Explain your findings. Study other values of η . What about $\eta = 10$?

- b) Investigate the following map within the context of universal behaviour

$$x_{n+1} = 3.5r(1 - r) \sin(\pi x_n) \quad x_0 \in [0, 1] \quad r \in [0, 1]$$

Discuss your results.

2.9.4 Discrete Cubic Map

Downloads: `DiscreteCubicMap.mws`

Consider the following discrete map

$$x_{n+1} = rx_n(3 - 4x_n^2) \tag{2.46}$$

with $x_0 \in [-1, 1]$ and $r \in [0, 1]$.

- a) Take $r = 0.97$. Generate a series of $N = 1000$ points starting from $x_0 = 0.2$. Show the series and the return-plot.
- b) Argue why r must be within $[-1, 1]$ to guarantee that all $x_n \in [-1, 1]$ when $x_0 \in [-1, 1]$.
- c) Find the fixed-points of (2.46) for $r \in [0, 1]$, and determine their stability.

d) Show that

$$x^{(2)} = \frac{\sqrt{2r(3r - \sqrt{9r^2 - 4})}}{4r}$$

is a period-2 solution. Find the region of r where this period-2 solution is stable.

e) Show a bifurcation diagram of the map over the range $r \in [0, 1]$.

f) Take $r = 1$ and set `Digits:=40`. Generate two series from slightly different initial conditions and estimate the Lyapunov-exponent.

We conjecture that for $r = 1$ the following closed form can be obtained:

$$x_n = \sin(3^n \theta) \tag{2.47}$$

where θ is such that $x_0 = \sin(\theta)$ is satisfied.

g) Show that (2.47) indeed satisfies (2.46) for $r = 1$. Does this imply that there is no predictability problem in this case? What can you say about the Lyapunov-exponent?

3 Discrete mappings of higher order

3.1 Fixed points and stability

A two-dimensional iterative map has the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned} \quad (3.1)$$

Fixed points can be determined by realizing that for large n : $x_{n+1} \rightarrow x_n \rightarrow x^*$, and $y_{n+1} \rightarrow y_n \rightarrow y^*$; hence the fixed point (x^*, y^*) satisfies

$$\begin{aligned} x^* &= f(x^*, y^*) \\ y^* &= g(x^*, y^*) \end{aligned} \quad (3.2)$$

If one finds solutions satisfying (3.2), one can determine the stability of the fixed point(s) by perturbing it slightly and studying the development, i.e. $x_n = x^* + \delta_n$, $y_n = y^* + \epsilon_n$ with $|\delta_n| \ll 1$, $|\epsilon_n| \ll 1$. Employing a Taylor series approximation

$$\begin{aligned} x^* + \delta_{n+1} &= f(x^* + \delta_n, y^* + \epsilon_n) \\ &= f(x^*, y^*) + \frac{\partial f}{\partial x} \delta_n + \frac{\partial f}{\partial y} \epsilon_n + \mathcal{O}(\delta^2, \epsilon^2) \\ y^* + \epsilon_{n+1} &= g(x^* + \delta_n, y^* + \epsilon_n) \\ &= g(x^*, y^*) + \frac{\partial g}{\partial x} \delta_n + \frac{\partial g}{\partial y} \epsilon_n + \mathcal{O}(\delta^2, \epsilon^2) \end{aligned}$$

Using (3.2) and neglecting terms of order δ^2, ϵ^2 one arrives at the equations governing the deviations from the fixed-point:

$$\begin{aligned} \delta_{n+1} &= \frac{\partial f}{\partial x}(x^*, y^*) \delta_n + \frac{\partial f}{\partial y}(x^*, y^*) \epsilon_n \\ \epsilon_{n+1} &= \frac{\partial g}{\partial x}(x^*, y^*) \delta_n + \frac{\partial g}{\partial y}(x^*, y^*) \epsilon_n \end{aligned} \quad (3.3)$$

We write this into a more formal form, by introducing the Jacobian $J(x, y)$

$$J(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

Hence with $J^* = J(x^*, y^*)$, the Jacobian in the fixed point, ((3.3)) can be cast into

$$\begin{pmatrix} \delta_{n+1} \\ \epsilon_{n+1} \end{pmatrix} = J^* \begin{pmatrix} \delta_n \\ \epsilon_n \end{pmatrix}$$

Stability of the fixed point requires that deviations δ_n, ϵ_n vanish as n increases. Whether this is the case will depend on the *eigenvalues* of J^* . Let S be the (invertible) matrix that reduces J^* to its diagonal form $\tilde{J}^*$ (we disregard the exceptional situations when it is not possible to diagonalize J^*):

$$\tilde{J}^* = S^{-1} J^* S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

where $\sigma_{1,2}$ denote the eigenvalues. Denoting the deviations in the diagonal system by $\tilde{\delta}_n, \tilde{\epsilon}_n$

$$\begin{pmatrix} \tilde{\delta}_n \\ \tilde{\epsilon}_n \end{pmatrix} = S^{-1} \begin{pmatrix} \delta_n \\ \epsilon_n \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\delta}_{n+1} \\ \tilde{\epsilon}_{n+1} \end{pmatrix} = \tilde{J}^* \begin{pmatrix} \tilde{\delta}_n \\ \tilde{\epsilon}_n \end{pmatrix} = \begin{pmatrix} \sigma_1 \tilde{\delta}_n \\ \sigma_2 \tilde{\epsilon}_n \end{pmatrix}$$

one observes that the system decouples, meaning that $\tilde{\delta}_n$ and $\tilde{\epsilon}_n$ do not affect each other's evolution. Using the exact solution we can express the evolution of the deviations as a function of the initial deviations and the eigenvalues

$$\tilde{\delta}_n = \sigma_1^n \tilde{\delta}_0 \quad \tilde{\epsilon}_n = \sigma_2^n \tilde{\epsilon}_0$$

Requiring in case of stability that both $\tilde{\delta}_n \rightarrow 0$ and $\tilde{\epsilon}_n \rightarrow 0$, implies that the *absolute value* of both eigenvalues be smaller than 1. Hence

$$(x^*, y^*) \text{ is stable when: } |\sigma_1| < 1 \text{ and } |\sigma_2| < 1$$

Note that the eigenvalues $\sigma_{1,2}$ can be complex valued.

3.2 Higher order mappings

One can readily generalize the results of the previous section to mappings of arbitrary order m :

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \tag{3.4}$$

where both $\mathbf{x}$ and $\mathbf{f}$ are m -dimensional vectors. A fixed point satisfies

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*) \tag{3.5}$$

Its stability follows from evaluating the $m \times m$ dimensional Jacobian matrix

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} \tag{3.6}$$

in the fixed point $\mathbf{x}^*$, i.e. $J^* = J(\mathbf{x}^*)$, and subsequently calculating its m eigenvalues σ_i , $i = 1, \dots, m$. The fixed point is stable when the absolute value of all eigenvalues is smaller than unity

$$\forall_{i \in \{1, \dots, m\}} |\sigma_i| < 1$$

Period 2 solutions can be found by realizing that they are fixed points of the mapping between second iterates, i.e.: $\mathbf{x}_{n+2} = \mathbf{x}_n = \mathbf{x}^{**}$. Hence one must look for solutions that satisfy

$$\mathbf{x}^{**} = \mathbf{g}(\mathbf{x}^{**}) \quad \text{with} \quad \mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{f}(\mathbf{x}))$$

while keeping in mind that from the resulting solution set one must exclude period-1 solutions which satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$ and therefore also satisfy $\mathbf{f}(\mathbf{f}(\mathbf{x}^*)) = \mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$. Stability of a period 2 solution can be determined in the same way as for period 1 solutions, that is, by calculating the Jacobian of $\mathbf{g}$ in the period-2 solution $\mathbf{x}^{**}$, determining the eigenvalues σ_i and checking whether or not all $|\sigma_i| < 1$.

3.2.1 Example: calculating fixed points and their stability for the Duffing map

The so-called Duffing map is given by

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= -bx_n + ay_n - y_n^3 \end{aligned} \quad (3.7)$$

with a, b parameters. See Figure 3.1 for a few time series and phase space plots, i.e. plots of (x_n, y_n) . They were made with the following `maple` program

 **Maple**

```
restart; with(plots):
f1 := (x, y) -> y;
f2 := (x, y) -> -b*x+a*y-y^3;
a := 2.52; b := 0.1;
N:=5000;
X := Array(0..N): X[0] := 1.0:
Y := Array(0..N): Y[0] := 1.0:
for n from 0 to N-1 do
    X[n+1] := evalf( f1(X[n], Y[n]) );
    Y[n+1] := evalf( f2(X[n], Y[n]) );
end do:
pointplot([seq([n, X[n]], n=0..N)], color=black,
           symbol=point, labels=["n", "x[n]"]);
pointplot([seq([X[n], Y[n]], n=0..N)], color=black,
           symbol=point, labels=["x[n]", "y[n]"]);
```

The map can be cast into the generic form (3.4) by defining

$$\begin{aligned} f_1(x, y) &= y \\ f_2(x, y) &= -bx + ay - y^3 \end{aligned}$$

Fixed points (x^*, y^*) follow from solving (3.5), which in this case amounts to solving

$$x^* = y^* \quad y^* = -by^* + ay^* - y^{*3}$$

This yields three fixed points

$$\mathbf{x}_1^* = (0, 0) \quad \mathbf{x}_2^* = (\sqrt{a-b-1}, \sqrt{a-b-1}) \quad \mathbf{x}_3^* = -\mathbf{x}_2^*$$

To determine the stability of these fixed points, we first determine the Jacobian (3.6) for this map:

$$J(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ -b & a - 3y^2 \end{pmatrix}$$

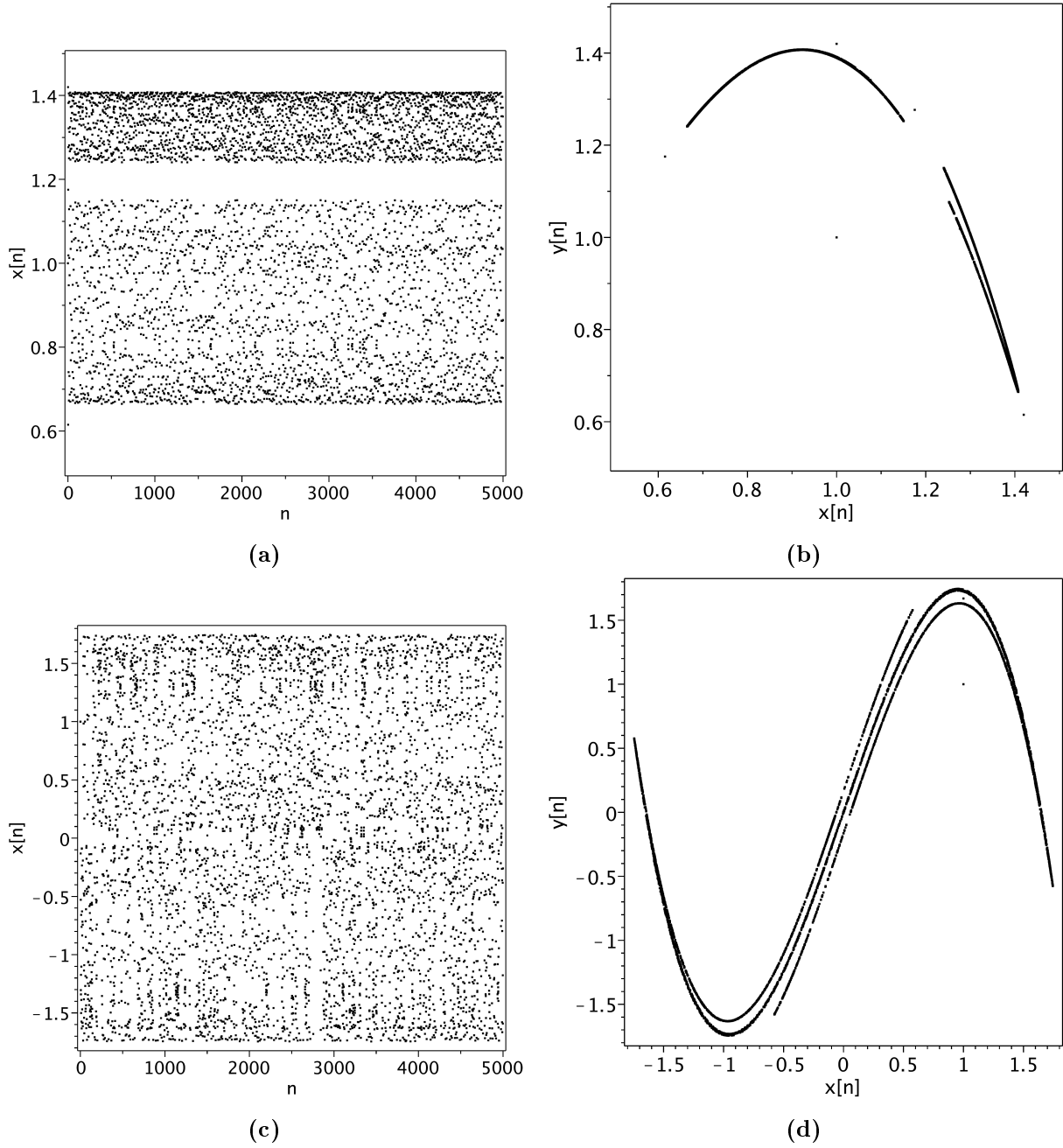


Figure 3.1: Series and phase space plots of the Duffing map (3.7) with $b = 0.1$. (a) Series of x_n for $a = 2.52$. (b) Phase plot (x_n, y_n) for $a = 2.52$. (c) Series x_n for $a = 2.77$. (d) Phase plot (x_n, y_n) for $a = 2.77$.

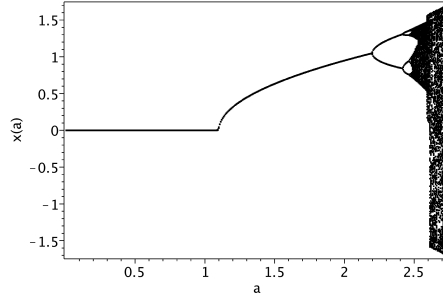


Figure 3.2: Bifurcation diagram of the Duffing map (3.7) as a function of a . $b = 0.1$ is fixed.

Substitution of the fixed points gives

$$J(\mathbf{x}_1^*) = \begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix} \quad J(\mathbf{x}_2^*) = \begin{pmatrix} 0 & 1 \\ -b & -3b + 3 - 2a \end{pmatrix}$$

The third fixed point yields the same matrix as the second fixed point ($J(\mathbf{x}_2^*) = J(\mathbf{x}_3^*)$) and therefore has exactly the same stability characteristics as $\mathbf{x}_2^*$. The corresponding eigenvalues are

$$\begin{aligned} \sigma_{\pm}(\mathbf{x}_1^*) &= \frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b} \\ \sigma_{\pm}(\mathbf{x}_2^*) &= \frac{1}{2}(3b + 3 - 2a) \pm \frac{1}{2}\sqrt{(3b + 3 - 2a)^2 - 4b} \end{aligned}$$

which reveals the stability of the fixed points for arbitrary a, b . Taking b fixed and solving a from $|\sigma_{\pm}| = 1$, gives $a = 1 + b$ as the critical value for $(0, 0)$ to become unstable. Repeating the analysis for the second fixed point shows that it is stable for a in the interval $[1 + b, 2 + 2b]$. By making a bifurcation diagram one can check the predictions, see Figure 3.2 for the case of $b = 1/10$; indeed the transitions are found at $a = 1.1$ and $a = 2.2$.

It also possible to calculate the period-2 solutions and their stability range. To this end we first define the mapping associated with the second iterates: $x_{n+2} = g_1(x_n, y_n)$, $y_{n+2} = g_2(x_n, y_n)$

$$\begin{aligned} g_1(x, y) &= f_1(f_1(x, y), f_2(x, y)) \\ g_2(x, y) &= f_2(f_1(x, y), f_2(x, y)) \end{aligned}$$

A period-2 solution to mapping $\mathbf{f}$ is a fixed point to the mapping $\mathbf{g}$. So we search for solutions to $\mathbf{x}^{**} = \mathbf{g}(\mathbf{x}^{**})$. The stability of these solution follows from determining the Jacobian of $\mathbf{g}$, substituting the fixed point solution to get J^* and calculating when the corresponding eigenvalues are ± 1 . This is what is carried out by the following `maple` implementation

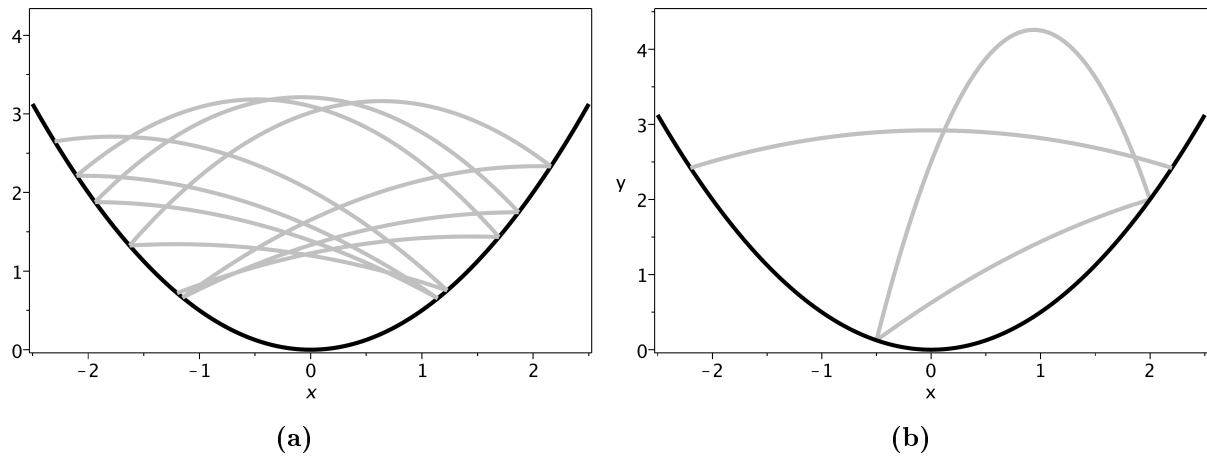


Figure 3.3: Trajectories of a bouncing ball in a parabolic bowl. (a) First 10 bounces for $X_0 = -1.2, U_0 = 2.2, V_0 = 1.2$. (b) Two examples of a periodic solution.



Maple

```
restart; with(plots): with(LinearAlgebra):
f1 := (x, y) -> y;
f2 := (x, y) -> -b*x+a*y-y^3;
g1 := (x, y) -> f1(f1(x,y),f2(x,y));
g2 := (x, y) -> f2(f1(x,y),f2(x,y));
sol:=solve({x=g1(x,y),y=g2(x,y)},{x,y});
fp2:=allvalues(sol[4]);
Jac:= VectorCalculus:-Jacobian([g1(x,y), g2(x,y)], [x,y]);
Jac1:= subs(fp2[1],Jac);
ev1:=Eigenvalues(Jac1);
solve(ev1[1]=1,a); solve(ev1[1]=-1,a);
solve(ev1[2]=1,a); solve(ev1[2]=-1,a);
```

It reveals that the period-2 solution is stable for a in the interval $[2b + 2, \sqrt{5b^2 + 8b + 5}]$. For $b = 1/10$ this amounts to approximately $[2.2, 2.419]$, which nicely corresponds to the behaviour seen in the bifurcation diagram in Figure 3.2. Note in the `maple` program the period-1 solutions, which also satisfy $\mathbf{x}^* = \mathbf{g}(\mathbf{x}^*)$, are disregarded by ignoring the first three solutions. A closer look at `sol[4]` reveals that it contains two families of period-2 solutions (so four entries in total), because of the symmetry in the map: If $\mathbf{x}^{**} = \mathbf{g}(\mathbf{x}^{**})$, then also $-\mathbf{x}^{**} = \mathbf{g}(-\mathbf{x}^{**})$ is a solution.

3.2.2 Example: a bouncing ball in a bowl

Consider a ball bouncing in a parabolically¹ shaped bowl, see Figure 3.3. The bowl is parametrized by

$$y_b(x) = \frac{x^2}{2\ell} \quad (3.8)$$

¹ This is a somewhat simpler version of a ball bouncing in a wedge shaped confinement

in which the parameter ℓ is a measure of the concavity of the bowl. Both ℓ , x and y have dimension meter. As the ball moves under the influence of gravity, the equations of motion are given by

$$\begin{aligned} x(t) &= x_0 + u_0 t & u(t) &= u_0 \\ y(t) &= y_0 + v_0 t - \frac{1}{2}gt^2 & v(t) &= v_0 - gt \end{aligned} \quad (3.9)$$

where x_0 and y_0 are the initial horizontal and vertical position, respectively; u_0 and v_0 are the corresponding initial velocities. Time can be eliminated from the problem using the first equation, giving

$$y(x) = y_0 + v_0 \left(\frac{x - x_0}{u_0} \right) - \frac{1}{2}g \left(\frac{x - x_0}{u_0} \right)^2 \quad (3.10)$$

We wish to determine the free fall trajectory between the previous bounce location (x_0, y_0) and the next one (x_1, y_1) . From (3.8) we know that both $y_0 = x_0^2/(2\ell)$ and $y_1 = x_1^2/(2\ell)$, hence using (3.10) we can solve x_1 from

$$\frac{x_1^2}{2\ell} = \frac{x_0^2}{2\ell} + v_0 \left(\frac{x_1 - x_0}{u_0} \right) - \frac{1}{2}g \left(\frac{x_1 - x_0}{u_0} \right)^2$$

Apart from the trivial solution $x_1 = x_0$ this yields

$$x_1 = \frac{2\ell u_0 v_0 + x_0 (g\ell - u_0^2)}{g\ell + u_0^2} \quad (3.11)$$

which in addition gives the velocities just before the bounce

$$u_1 = u_0 \quad v_1 = v_0 - g \left(\frac{x_1 - x_0}{u_0} \right) \quad (3.12)$$

The collision is elastic, which implies that the wall-normal velocity exactly changes sign. Using vector notation $\mathbf{u} = (u, v)$, one can decompose the velocity vector into a wall-normal and a parallel component $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$. Let us denote velocity fields before the bounce by $\mathbf{u}^-$ and after the bounce by $\mathbf{u}^+$. In this notation we have $\mathbf{u}_\parallel^+ = \mathbf{u}_\parallel^-$, $\mathbf{u}_\perp^+ = -\mathbf{u}_\perp^-$. This yields the velocity vector after the bounce

$$\mathbf{u}^+ = \mathbf{u}_\parallel^+ + \mathbf{u}_\perp^+ = \mathbf{u}_\parallel^- - \mathbf{u}_\perp^- = \mathbf{u}^- - 2\mathbf{u}_\perp^-$$

Denoting the wall-normal vector (normalized vector perpendicular to the wall) by $\mathbf{n}$ allows one to express $\mathbf{u}_\perp^- = \mathbf{n}(\mathbf{n} \cdot \mathbf{u}^-)$, where $(\mathbf{n} \cdot \mathbf{u}^-)$ is the inner product of $\mathbf{n}$ and $\mathbf{u}^-$. So for the n -th bounce

$$\mathbf{u}_n^+ = \mathbf{u}_n^- - 2\mathbf{n}(\mathbf{n} \cdot \mathbf{u}_n^-) \quad (3.13)$$

For the parabolic bowl (3.8) the normal vector at x_n is given by

$$\mathbf{n} = \frac{1}{\sqrt{x_n^2 + \ell^2}} \begin{pmatrix} x_n \\ -\ell \end{pmatrix}$$

so with (3.13) we get

$$\begin{aligned} u_n^+ &= \frac{\ell^2 - x_n^2}{\ell^2 + x_n^2} u_n^- + \frac{2\ell x_n}{\ell^2 + x_n^2} v_n^- \\ v_n^+ &= \frac{2\ell x_n}{\ell^2 + x_n^2} u_n^- - \frac{\ell^2 - x_n^2}{\ell^2 + x_n^2} v_n^- \end{aligned}$$

Equations (3.12) and (3.11) relate (x_n, u_n^-, v_n^-) to $(x_{n-1}, u_{n-1}^+, v_{n-1}^+)$, so we are now able to relate successive bounce events $(x_n, u_n^+, v_n^+) \rightarrow (x_{n+1}, u_{n+1}^+, v_{n+1}^+)$ by the following set of equations

$$\begin{aligned} x_{n+1} &= \frac{2\ell u_n v_n + x_n (g\ell - u_n^2)}{g\ell + u_n^2} \\ u_{n+1} &= \frac{\ell^2 - x_{n+1}^2}{\ell^2 + x_{n+1}^2} u_n + \frac{2\ell x_{n+1}}{\ell^2 + x_{n+1}^2} \left[v_n - g \left(\frac{x_{n+1} - x_n}{u_n} \right) \right] \\ v_{n+1} &= \frac{2\ell x_{n+1}}{\ell^2 + x_{n+1}^2} u_n - \frac{\ell^2 - x_{n+1}^2}{\ell^2 + x_{n+1}^2} \left[v_n - g \left(\frac{x_{n+1} - x_n}{u_n} \right) \right] \end{aligned} \quad (3.14)$$

where the $+$ -symbols on top of the velocities have now been omitted.

It would seem that the system has two parameters, g , and ℓ . However, upon introducing the non-dimensional variables $X_n = x_n/\ell$, $U_n = u_n/\sqrt{g\ell}$, $V_n = v_n/\sqrt{g\ell}$, these parameters g and ℓ drop out of the equations, revealing the essence of the mapping:

$$\begin{aligned} X_{n+1} &= \frac{2U_n V_n + X_n (1 - U_n^2)}{1 + U_n^2} \\ U_{n+1} &= \frac{1 - X_{n+1}^2}{1 + X_{n+1}^2} U_n + \frac{2X_{n+1}}{1 + X_{n+1}^2} \left[V_n - \left(\frac{X_{n+1} - X_n}{U_n} \right) \right] \\ V_{n+1} &= \frac{2X_{n+1}}{1 + X_{n+1}^2} U_n - \frac{1 - X_{n+1}^2}{1 + X_{n+1}^2} \left[V_n - \left(\frac{X_{n+1} - X_n}{U_n} \right) \right] \end{aligned} \quad (3.15)$$

which shows that only the initial conditions X_0, U_0, V_0 are relevant for the system. Using the dimensional form (3.14) with g and ℓ would only trivially rescale the outcome.

Below find a `maple` implementation of the bouncing ball, which gives a picture as shown in Figure 3.3.



Maple

```
restart; with(plots):
yb := x->x^2/2;
y := (x,x0,u0,v0)->yb(x0)+v0*(x-x0)/u0-1/2*(x-x0)^2/u0^2;
N := 10;
X := Array(0..N): Y := Array(0..N): U := Array(0..N): V := Array(0..N):
X[0] := -1.2; U[0] := 2.2; V[0] := 1.2;
for n from 0 to N-1 do
  X[n+1] := (2*U[n]*V[n]+X[n]*(1-U[n]^2))/(1+U[n]^2);
  vtmp := V[n] - (X[n+1]-X[n])/U[n];
  U[n+1] := ((1-X[n+1]^2)*U[n] + 2*X[n+1]*vtmp)/(1+X[n+1]^2);
  V[n+1] := (2*X[n+1]*U[n] - (1-X[n+1]^2)*vtmp)/(1+X[n+1]^2);
  pl[n] := plot(y(x,X[n],U[n],V[n]),x=X[n]..X[n+1]);
end do;
plf := plot(yb(x),x=-1..1);
display(plf,seq(pl[n],n=0..N-1),labels=["x","y"],axes=framed);
```

As a result of the elastic collisions, the system will conserve energy. In Figure 3.4 we have plotted the (dimensionless) kinetic and potential energy at the bounces $E_{k,n} = \frac{1}{2}[U_n^2 + V_n^2]$, $E_{p,n} = Y_n = \frac{1}{2}X_n^2$. In addition we have indicated the sum, which is indeed conserved for all n , i.e.

$$E_{t,n} = E_{k,n} + E_{p,n} = E_{k,0} + E_{p,0}$$

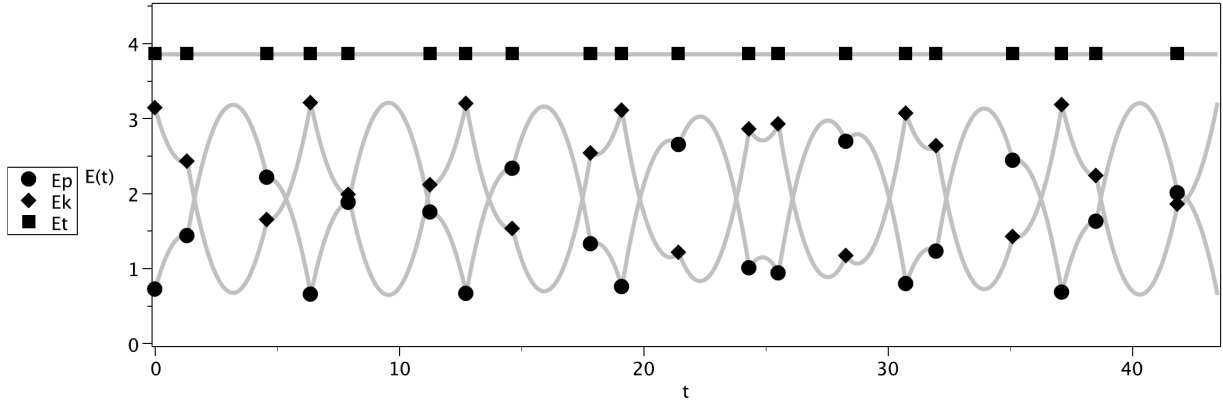


Figure 3.4: Evolution in time of the potential energy $E_{p,n}$ (circles), kinetic energy $E_{k,n}$ (diamonds) and the sum of both $E_{t,n}$ (boxes). Time t was 'recreated' using (3.9), allowing the values between bounces to be displayed as well.

Does the system (3.15) allow stationary (periodic) situations? There is a period-1 solution possible where the ball hops up and down precisely in the center with no horizontal velocity: $X^* = Y^* = 0$. One can obtain infinitely many period-2 solutions when $v_0 = 1$:

$$(x_0, u_0, v_0) = (-a, a, 1) \rightarrow (a, -a, 1) \rightarrow (-a, a, 1) \rightarrow$$

with arbitrary a . With a little more effort one can also find less obvious solutions such as the asymmetric periodic trajectory

$$(x_0, u_0, v_0) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{23}{8}\right) \rightarrow \left(2, -2, -\frac{7}{8}\right) \rightarrow \left(-\frac{1}{2}, \frac{1}{2}, \frac{23}{8}\right) \rightarrow$$

Both periodic solutions are shown in Figure 3.3.

3.3 Growth of errors revisited: Lyapunov exponents of higher order mappings

Consider two data-sets resulting from the same mapping $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, $\mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n)$ but resulting from slightly differing initial conditions, i.e. $\mathbf{y}_n = \mathbf{x}_0 + \boldsymbol{\epsilon}_0$ with $\|\boldsymbol{\epsilon}_0\| \ll 1$. Denoting the difference between the two data-sets at step n as $\boldsymbol{\epsilon}_n$ and assuming the differences are still small, one can derive (see section 3.1)

$$\begin{aligned} \boldsymbol{\epsilon}_{n+1} &= \mathbf{y}_{n+1} - \mathbf{x}_{n+1} \\ &= \mathbf{f}(\mathbf{x}_n + \boldsymbol{\epsilon}_n) - \mathbf{f}(\mathbf{x}_n) \\ &= J(\mathbf{x}_n)\boldsymbol{\epsilon}_n \end{aligned}$$

where $J(\mathbf{x}_n)$ is the Jacobian matrix (3.6) evaluated at data-point $\mathbf{x}_n$. Using the above expression recursively until $n = 0$, one obtains

$$\boldsymbol{\epsilon}_n = J(\mathbf{x}_{n-1})J(\mathbf{x}_{n-2}) \dots J(\mathbf{x}_1)J(\mathbf{x}_0)\boldsymbol{\epsilon}_0 \quad (3.16)$$

It is important to note that these matrices might all be different because they are evaluated at different $\mathbf{x}_n$ and probably will not commute, i.e. $J(\mathbf{x}_n)J(\mathbf{x}_{n-1}) \neq J(\mathbf{x}_{n-1})J(\mathbf{x}_n)$, which is to say

that they do not acquire their diagonal form by the same transformation S . To proceed let us recapitulate how Lyapunov exponents Λ were derived in one-dimensional mappings (see (2.24) and (2.25)). Rewriting slightly, we get

$$\bar{\sigma}^N = f'(x_0)f'(x_1)f'(x_2)\dots f'(x_{N-1}) \rightarrow \Lambda = \log|\bar{\sigma}| \quad (3.17)$$

showing that Lyapunov exponent Λ is the log of $|\bar{\sigma}|$, where $\bar{\sigma}$ can be viewed as the *geometric average* of the derivatives $f'(x_n)$, $n = 0 \dots N-1$. In the same vein we can think of an average Jacobian $\bar{J}^n$ in (3.16)

$$\bar{J}^N = J(\mathbf{x}_{N-1})J(\mathbf{x}_{N-2})\dots J(\mathbf{x}_1)J(\mathbf{x}_0)$$

If we denote the (possibly complex valued) eigenvalues of $\bar{J}$ by $\bar{\sigma}_1, \dots, \bar{\sigma}_m$, then the eigenvalues of $\bar{J}^N$ are $\bar{\sigma}_1^N, \dots, \bar{\sigma}_m^N$. Conversely, if one has calculated the eigenvalues of $\bar{J}^N$, then the eigenvalues of $\bar{J}$ are found by taking the N -th root. When the eigenvalues $\bar{\sigma}_1, \dots, \bar{\sigma}_m$ are known, then by analogy with (3.17) we can express the Lyapunov exponents as

$$\Lambda_1 = \log|\bar{\sigma}_1|, \quad \Lambda_2 = \log|\bar{\sigma}_2|, \quad \dots \quad \Lambda_m = \log|\bar{\sigma}_m| \quad (3.18)$$

One observes that an m -dimensional mapping has m Lyapunov exponents.

In short, a (formal) procedure for finding the m Lyapunov exponents of a mapping $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ consists of (i) creating a series $\mathbf{x}_n$, $n = 0 \dots N-1$, (ii) calculating $\bar{J}^N$ and determining its eigenvalues $\bar{\sigma}^N$, (iii) applying (3.18).

A more practical way for getting information on the sensitivity on initial conditions of a certain mapping would be to create two data-sets $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, $\mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n)$ started from slightly different initial conditions $\mathbf{y}_0 = \mathbf{x}_0 + \boldsymbol{\epsilon}_0$, and monitoring the evolution of the distance $\delta_n = \|\boldsymbol{\epsilon}_n\| = \sqrt{\boldsymbol{\epsilon}_n \cdot \boldsymbol{\epsilon}_n}$. One anticipates a dependence of the form

$$\delta_N = \delta_0 \exp(\Lambda_e N)$$

where Λ_e can be considered an *effective* Lyapunov exponent. But what is the relation between Λ_e and the 'formal' Lyapunov exponents of (3.18)?

Using (3.16) we see that

$$e^{2\Lambda_e N} = \frac{\delta_N^2}{\delta_0^2} = \frac{\boldsymbol{\epsilon}_N \cdot \boldsymbol{\epsilon}_N}{\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0} = \frac{\bar{J}^N \boldsymbol{\epsilon}_0 \cdot \bar{J}^N \boldsymbol{\epsilon}_0}{\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0}$$

If $\boldsymbol{\epsilon}_0$ is exactly aligned with the k -th eigenvector of $\bar{J}$, which corresponds to the eigenvalue $\bar{\sigma}_k$, then $\bar{J}^N \boldsymbol{\epsilon}_0 = \bar{\sigma}_k^N \boldsymbol{\epsilon}_0$, and therefore

$$e^{2\Lambda_e N} = \bar{\sigma}_k^{2N} = e^{2\Lambda_k N} \quad \rightarrow \quad \Lambda_e = \Lambda_k \quad k \in [1, \dots, m]$$

Ordering the Lyapunov exponents from small to large, we can write for the effective Lyapunov exponent

$$\Lambda_1 \leq \Lambda_e \leq \Lambda_m$$

showing that the 'measured' effective Lyapunov exponent will always be between the values obtained from (3.18). But it is incorrect to assume that Λ_e will be an average of the $\Lambda_1, \dots, \Lambda_m$.

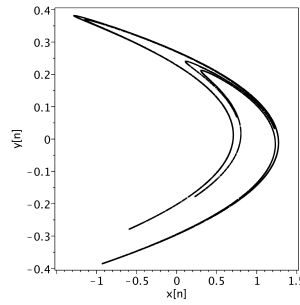


Figure 3.5: The Hénon attractor for $a = 1.4$, $b = 0.3$.

3.3.1 Example: Lyapunov exponents of the Hénon map

The Hénon map is given by $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ with

$$\begin{aligned} f_1(x, y) &= y + 1 - ax^2 \\ f_2(x, y) &= bx \end{aligned} \tag{3.19}$$

where $a \in [0, 1.5]$ and $b \in [0, 1]$ are adjustable parameters. The Jacobian (3.6) for this map is

$$J(\mathbf{x}) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

In Figure 3.5 we show the classical result in phase space (x_n, y_n) for the parameter choice of $a = 1.4$ and $b = 0.3$. To obtain such a graph one can use the same `maple` code as used for the Duffing map in section 3.2.1. Below we set out to obtain the Lyapunov exponents (3.18) for the Hénon map with $a = 1.4$ and $b = 0.3$. See the `maple` code. First we define the mapping and determine the Jacobian; using the `unapply` command we can later easily pass the arguments via a call to $J(X[n], Y[n])$. Next we iterate the mapping $N = 400$ times, each time performing the matrix multiplication with $J(\mathbf{x}_n)$ and storing the cumulative result in $\mathbf{C}$, which in the end therefore contains $\bar{J}^N$. In the next steps the eigenvalues are calculated, from which the Lyapunov exponents are derived using (3.18). The `maple` code below produces something like $\Lambda_1 \approx -1.6$, $\Lambda_2 \approx +0.4$. It is important to note that one Lyapunov exponent is positive and one is negative. This explains why we used a very large number of digits in the code. This is crucial because one eigenvalue rapidly decreases as $\exp(\Lambda_1 N)$, while the other rapidly grows as $\exp(\Lambda_2 N)$; this obviously necessitates a high numerical precision to continue to perform the matrix multiplications with reasonable accuracy. Probably more sophisticated algorithms are available to perform the above procedure, but we chose to employ maple's feature to set `Digits` to a very large number and use a brute force method which has the advantage of being very straightforward.

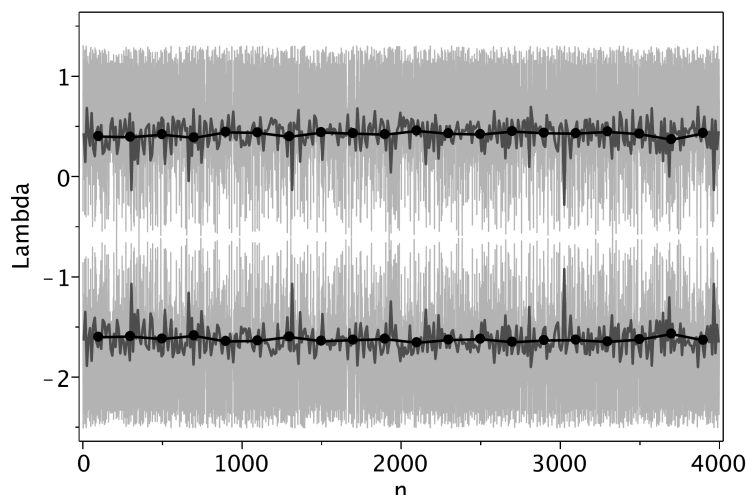


Figure 3.6: The two Lyapunov exponents of the Hénon map determined in three different ways: 1) black lines (and circles): calculated 20 times with $N = 200$, 2) dark gray lines: calculated 400 times with $N = 10$ and 3) light gray lines: calculated 4000 times with $N = 1$.



Maple

```
restart; with(plots): with(LinearAlgebra): with(VectorCalculus,Jacobian);
Digits := 800;
f1 := (x, y) -> y + 1 - a * x^2;
f2 := (x, y) -> b * x;
J := unapply(Jacobian([f1(x,y), f2(x,y)], [x,y]), [x,y]):
a := 1.4; b := 0.3;
N := 400;
X := Array(0..N): Y := Array(0..N):
X[0] := 0.1; Y[0] := 0.3;
C := Matrix([[1,0],[0,1]]);
for n from 0 to N-1 do
  X[n+1] := evalf( f1(X[n], Y[n]) );
  Y[n+1] := evalf( f2(X[n], Y[n]) );
  C := MatrixMatrixMultiply(J(X[n],Y[n]),C);
end do:
lam := Eigenvalues(C):
Lyap1 := log(min(abs(lam)))/N:
Lyap2 := log(max(abs(lam)))/N:
evalf[10](Lyap1); evalf[10](Lyap2);
```

One may wonder whether the obtained values for the Lyapunov exponents still depend on N and/or the initial conditions. To answer this we have put an extra loop around the above computation such that it is performed P times, taking the last data point of the previous computation as initial conditions for the next computation. The results are displayed in Figure 3.6: one notices the large scatter when N is small and the convergence to a constant value when N is large enough. Taking an average of the $N = 1$ results, tends to provide wrong estimates ($\Lambda_1 = -1.8$ and $\Lambda_2 = 0.6$ respectively, where the latter is about 50% too large).

The finding that one Lyapunov exponent is positive and one is negative is typical for chaotic systems. The positive Lyapunov exponent reveals the sensitive dependence on initial conditions and the precise value of Λ_2 thus quantifies this sensitivity. The negative exponent, on the other

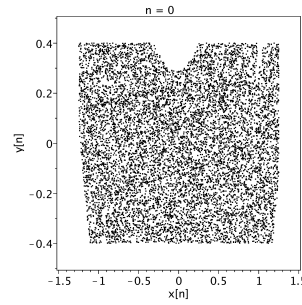


Figure 3.7: The action of *contraction* due to the negative Lyapunov exponent: shown are the first 5 steps of the evolution of 10^4 initial conditions which were randomly drawn from the interval $[-1.25, 1.25] \times [-0.4, 0.4]$.

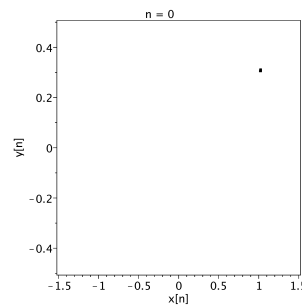


Figure 3.8: The action of *stretching* due to the positive Lyapunov exponent: the evolution is shown of 10^4 initial conditions very close to each other (max separation of 0.02).

hand, demonstrates that the mapping also exerts significant *contraction*; this explains why different initial conditions are *attracted* to the object – the attractor – displayed in Figure 3.5. The action of attraction can be nicely observed in Figure 3.7 where we have plotted the first 5 steps for a large number of random initial conditions (the 'holes' represent initial conditions that produce a divergent series and are omitted). The effect of *stretching* due to the positive Lyapunov exponent can be seen in Figure 3.8 where we have shown the evolution of the same number of initial conditions, which are now very close to each other.

3.4 Periodically kicked systems

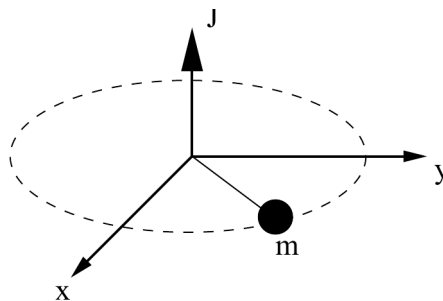


Figure 3.9: Sketch of a kicked rotor system

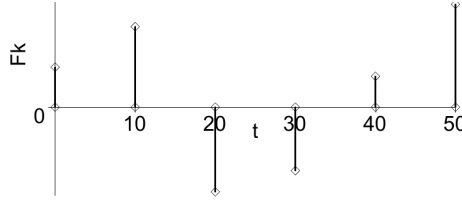


Figure 3.10: The dynamics of the undamped kicked rotor system

3.4.1 The damped kicked rotor: from continuous to discrete

The conceptually most straightforward example of a system with periodic kicking is the kicked rotor. Imagine a mass m attached to an arm of length r rotating in a horizontal plane (Figure 3.9). The system is not influenced by gravity, but is subject to linear friction. At fixed intervals T , the system is 'kicked': it receives a finite amount of angular momentum in a negligibly short time. The kicking strength I depends on the angle θ .

Before deriving the governing equations, let us analyse a typical timeseries of the kicked rotor system (Figure 3.10). The rotor is kicked at intervals $T = 10$ and for simplicity friction has been neglected. Figure 3.10 shows the impulsive forcing F_t due to the kicks, the radial velocity v_θ and the phase angle θ as a function of time. In between the kicks, no force acts on the rotor so the velocity stays constant. As a result, the angle θ changes linearly with time. At $t = 10$, the system is kicked (F_t shows the direction and relative magnitude of the forcing), by which the velocity changes instantaneously. The angle does not change in this short instance. As the kick at $t = 10$ is large and positive, the rotor speeds up (v_θ increases), which translates to an increase in the slope of θ . As the kicking strength I depends on the angle θ , each kick is generally of different strength. This property gives rise to some very complicated dynamics. But before we can explore this complexity, we need to develop the governing equations for the kicked rotor system.

The kicked rotor is governed by conservation of angular momentum J , which states that the rate-of-change of J is equal to the applied torque:

$$\dot{J} = \sum F_\theta r.$$

Here F_θ is the force in the x, y plane perpendicular to the arm. In between the kicks, the rotor is subject to linear friction $F_\theta = -\gamma v_\theta$, where v_θ is the radial velocity and γ a friction coefficient. The angular momentum J is defined as $J = mrv_\theta$ and $v_\theta = r\dot{\theta}$, so that the rotor (in absence of kicks) is governed by the following two linear ODE's:

$$\begin{aligned} r\dot{\theta} &= v_\theta, \\ m\dot{v}_\theta &= -\gamma v_\theta. \end{aligned} \tag{3.20}$$

Due to linearity, we can solve the system ((3.20)) without complications. Denoting the state of the system immediately after a kick by (θ_n^+, v_n^+) , the phase angle θ and radial velocity v_θ evolve according to

$$\begin{aligned} \theta(t) &= \frac{v_n^+ \tau}{r} (1 - e^{-t/\tau}) + \theta_n^+, \\ v_\theta(t) &= v_n^+ e^{-t/\tau}, \end{aligned} \tag{3.21}$$

where τ is the relaxation timescale $\tau = \frac{m}{\gamma}$.

Because we have the solution of the dynamics between the kicks, we can use ((3.21)) to 'fast forward' the state of the system to just before the next kick T seconds later, which we will denote by θ_{n+1}^- and v_{n+1}^- :

$$\begin{aligned}\theta_{n+1}^- &= \frac{v_n^+ \tau}{r} (1 - e^{-T/\tau}) + \theta_n^+, \\ v_{n+1}^- &= v_n^+ e^{-T/\tau}.\end{aligned}\tag{3.22}$$

The kick-relations, which connect the state of the system just before and after a kick are given by:

$$\begin{aligned}\theta_n^+ &= \theta_n^-, \\ v_n^+ &= v_n^- + V(\theta_n).\end{aligned}\tag{3.23}$$

Thus, the kick happens over such a short time that the angle does not change. However, the velocity changes by an amount of $V(\theta) = I(\theta)/mr$, where $I(\theta)$ is the angular momentum added to or extracted from the system. Using ((3.23)) to eliminate v_n^+ and θ_n^+ from ((3.22)), we obtain the following discrete two-dimensional map which governs the kicked rotor system:

$$\begin{aligned}\theta_{n+1}^- &= \frac{\tau}{r} (1 - e^{-T/\tau}) (v_n^- - V(\theta_n)) + \theta_n^- \\ v_{n+1}^- &= (v_n^- - V(\theta_n)) e^{-T/\tau}\end{aligned}\tag{3.24}$$

It is insightful to write this system in matrix format

$$\begin{pmatrix} \theta_{n+1}^- \\ v_{n+1}^- \end{pmatrix} = \begin{pmatrix} 1 & \frac{\tau}{r} (1 - e^{-T/\tau}) \\ 0 & e^{-T/\tau} \end{pmatrix} \begin{pmatrix} \theta_n^- \\ v_n^- \end{pmatrix} - \begin{pmatrix} \frac{\tau}{r} (1 - e^{-T/\tau}) \\ e^{-T/\tau} \end{pmatrix} V(\theta_n),$$

because this clearly elicits where the nonlinearity originates from. Note that the minus superscripts have been dropped. Note that the matrix is constant and therefore the operation is linear. The nonlinearity is purely contained in the phase-dependent kicking function $V(\theta)$.

3.4.2 Limit of $T \gg \tau$

From the general damped kicked rotor system ((3.24)), there are two important limit cases. The first limit is when the kicking interval T is much larger than the relaxation time τ , i.e. $T \gg \tau$. In this case, the system ((3.24)) reduces to

$$\begin{aligned}\theta_{n+1} &= \theta_n - \frac{\tau}{r} V(\theta_n), \\ v_{n+1} &= 0.\end{aligned}$$

Clearly, $v_{n+1} = 0$ holds for all n (as $T \gg \tau$, the rotor has come to a halt before the next kick). Hence, the system is governed by the following one-dimensional discrete map:

$$\theta_{n+1} = \theta_n - \frac{\tau}{r} V(\theta_n).$$

When we take for $V(\theta)$ a function of the form $V(\theta) = A + B \sin \theta$ and normalise θ and v appropriately we obtain the *circle map*. This map will be discussed in detail in section 3.4.5.

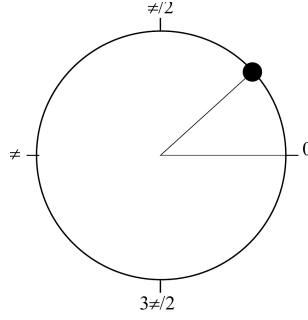


Figure 3.11: Sketch of a flow on a circle.

3.4.3 Limit of $\tau \rightarrow \infty$

The second important map can be obtained from ((3.24)), by considering the limit to an undamped rotor system. When the friction coefficient $\gamma \rightarrow 0$, the relaxation timescale $\tau = m/\gamma \rightarrow \infty$. Using that

$$\lim_{\tau \rightarrow \infty} \tau(1 - e^{-T/\tau}) = \lim_{\tau \rightarrow \infty} \tau \left(1 - \left[1 - \frac{T}{\tau} + \mathcal{O}\left(\frac{T^2}{\tau^2}\right) \right] \right) = T,$$

the system ((3.24)) simplifies to

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{T}{r} v_{n+1}, \\ v_{n+1} &= v_n - V(\theta_n). \end{aligned}$$

When we take for $V(\theta)$ a sinusoidally varying function $I(\theta) = A \sin \theta$ and normalise θ and v appropriately, we obtain the *standard map*. This map will be discussed in detail in section 3.4.6.

3.4.4 Linear maps on the circle

In this section, we will explore mappings defined on circular domains (Figure 3.11). Although these mappings can be treated with the same tools we used before, there are some important new features due to the different topology (moving from a line to a circle). The difference is demonstrated by the seemingly trivial linear map:

$$x_{n+1} = f(x_n) = x_n + \Omega,$$

where $\Omega > 0$ is a parameter. This linear equation has solution $x_n = x_0 + n\Omega$, and does not allow fixed points $x_{n+1} = x_n = x^*$ except when $\Omega = 0$.

However, fixed points become possible when we consider the same equation on a circular domain of unity size, given by

$$x_{n+1} = x_n + \Omega \mod 1 \tag{3.25}$$

Now x is a variable associated with a phase angle θ by $x = \theta/(2\pi)$ (Figure 3.11). The parameter Ω determines how fast the system rotates. In terms of the kicked rotor (Figure 3.9), (3.25) corresponds to a constant rotation number rate of the rotor which is sampled periodically. If we determine the fixed points of (3.25) we see that period-1 solutions occur when

$$\Omega = 0 \mod 1 \quad (\text{or } \Omega = 0, 1, 2, \dots).$$

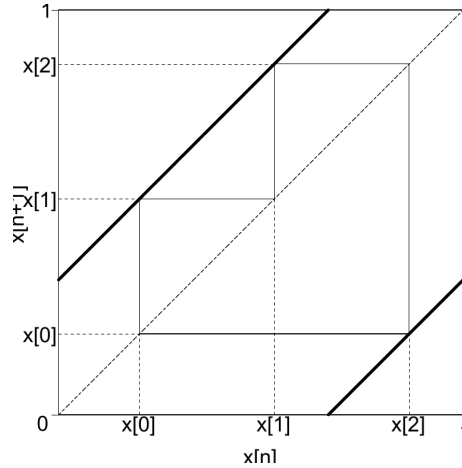


Figure 3.12: Cobweb diagram for a period-3 solution of the linear map (3.25).

Thus, this result shows that when Ω is equal to an arbitrary number of full rotation numbers, the system will have a period-1 solution. Note that the absence of x in the solution shows that this result is valid for any x_0 .

Let us now generalize this result to a solution of period m . Applying (3.25) m times, we obtain

$$x_{n+m} = f^{(m)}(x_n) = x_n + m\Omega \mod 1.$$

Requiring that $x_{n+m} = x_n = x^*$ shows that a period- m solution occurs when

$$\Omega = 0 \mod 1/m \quad (\text{or } \Omega = 0, \frac{1}{m}, \frac{2}{m}, \dots). \quad (3.26)$$

As was the case for determining higher order fixed points (section 2.3), lower order periodic solutions have to be excluded from (3.26). For example, period-3 solutions occur for $\Omega \in \{0, \frac{1}{3}, \frac{2}{3}, 1, \dots\}$, but one has to exclude the period-1 contributions $\Omega \in \{0, 1, \dots\}$. Therefore, true period-3 solutions occur for $\Omega \in \{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \dots\}$.

In `maple`, the following fragment generates a period-3 solution of (3.25):

```

 Maple
restart;
mod1 := x -> (x- floor(x));
f := x -> x + Omega;
fmod1 := x -> mod1(f(x));
N := 40;
X := Array(0..N):
Omega := 1 / 3;
X[0] := 0.2;
for n from 0 to N-1 do
    X[n+1] := evalf( fmod1(X[n]) );
end do;
```

Figure 3.12 shows the cobweb diagram for the period-3 solution. Note that it is necessary to define a `mod1` function, as the built-in function `mod` is meant for integer arithmetic.

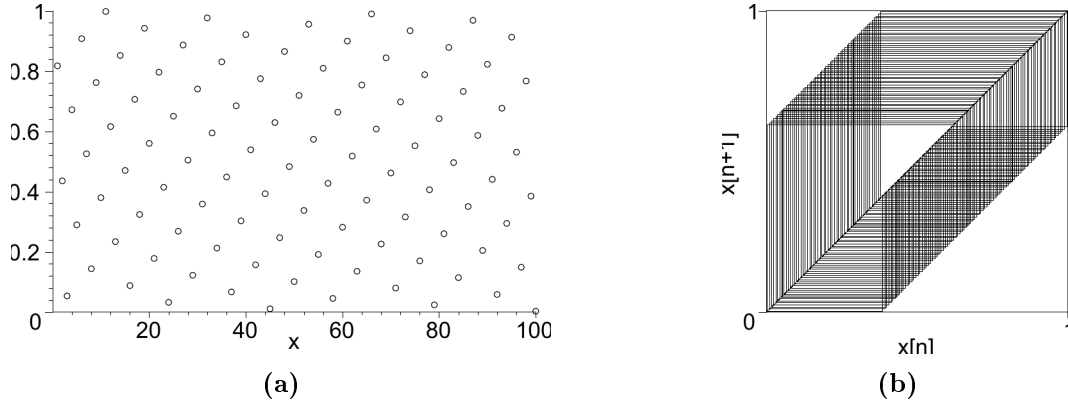


Figure 3.13: Quasi-periodicity of system (3.25) for $\Omega = 10.17$. (a) Series. (b) Cobweb.

Besides establishing the periodicity m of a solution, it is useful to define how many cycles R the system undergoes during these iterations. A periodic state then, can be uniquely characterised by the *rotation number* w defined by

$$w = \frac{\# \text{cycles}}{\# \text{iterations}} = \frac{R}{m}. \quad (3.27)$$

Denoting the map by $x_{n+1} = f(x_n)$, the number of cycles R can be calculated by $R = f^{(m)}(x_n) - x_n$. The mapping $f(x)$ should not contain the modulo operator, as it eliminates historical information like the number of cycles

For the linear map (3.25), $f^{(m)}(x_n) = x_n + m\Omega$, so that R is simply $R = m\Omega$.

As an example, we characterise the period-3 solutions $\Omega = \frac{1}{3}$ and $\Omega = \frac{2}{3}$. For $\Omega = \frac{1}{3}$, the system returns to its original position each three iterations ($m = 3$) after one revolution ($R = m\Omega = 1$), and thus $w = 1/3$. For $\Omega = \frac{2}{3}$, the system returns to its original position each three iterations ($m = 3$) but after two revolutions ($R = m\Omega = 2$), and thus $w = 2/3$. Thus, for the linear map, the rotation number w is identical to the value of Ω when the solution is periodic.

Intended to quantify periodic solutions on the circle, the rotation number w is by definition a rational number. However, the standard simplification rules of fractions do not apply to w . Take for example $w = 4/6$, which one may want to simplify to $w = 2/3$. This simplification is not allowed because $w = 4/6$ corresponds to a period-6 cycle and $w = 2/3$ to a period-3 cycle. If we could make this simplification, then the $w = 4/6$ case was not a period-6 solution in the first place, but two subsequent period-3 solutions.²

As follows from (3.26), there are an infinite number of cases for Ω with periodic solutions. Nevertheless, there are infinitely more cases for which the system is not periodic. That is, for all Ω that are *not* a fraction but an irrational number, the solution will not be periodic. Figure 3.13 shows the behavior of the system for $\Omega = \frac{1}{2}(1 + \sqrt{5})$. Although the iteration sequence shows that there is much order in the system, the cobweb diagram fills up as $n \rightarrow \infty$, indicating that the system traverses the entire domain. This is naturally a direct result of the choice of Ω : an irrational number cannot be converted to a fraction. Thus, no matter how many iterations and revolutions, the system will never return to its original location. A series like this is called *quasi-periodic*, as

² The state $w = 4/6$ does not exist for the linear map (3.25). The realisable period-6 solutions, after excluding the period 1, 2 and 3 solutions from (3.26), are $w \in \{1/6, 5/6\}$. However, for nonlinear maps a state like $w = 4/6$ can exist, because not all increments have the same size.

the space-filling behavior is a result of the mismatch between Ω and the size of the domain (which is 1).

Quasi-periodic and chaotic series share the ability never to return at the same location. Nevertheless, the two are fundamentally different. The difference resides in the sensitivity to initial conditions. The exact solution of (3.25) is given by $x_n = x_0 + n\Omega \pmod 1$. Introducing a second series y_n with slightly perturbed initial conditions $y_0 = x_0 + \epsilon_0$ and solution $y_n = y_0 + n\Omega \pmod 1$, the difference $y_n - x_n$ is given by

$$y_n - x_n = y_0 - x_0 = \epsilon_0.$$

The distance between the two series remains the same for all n , and we conclude that the system is not sensitive to initial conditions. This is a general feature of quasi-periodic solutions: even though never returning to the same location, a quasi-periodic solution lacks serious sensitivity to initial conditions. In terms of Lyapunov exponents, this situation corresponds to $\Lambda = 0$.

3.4.5 Phase-locking and the Devil's staircase

In the previous section the linear map (3.25) was shown to have an infinite number of periodic solutions. Nevertheless, there were infinitely many more solutions which were quasi-periodic. In this section we add a sinusoidal term to (3.25), thereby obtaining what is called the *circle map*

$$x_{n+1} = x_n + \Omega + \frac{K}{2\pi} \sin(2\pi x_n) \pmod 1. \quad (3.28)$$

This map can be derived from the kicked rotor when one assumes that the kicking interval is much larger than the time it takes for the system to stop rotating due to friction and adding a suitable forcing term (see section 3.4.1).

The question we set out to answer is how a non-linear sinusoidal term modifies the (quasi-)periodic behavior in (3.28) one would have for $K = 0$. One could surmise that the nonlinear term will destroy the periodic states and creates chaotic series. However, as we will see, instead of destroying periodic solutions, the nonlinear term in (3.28) does the opposite: it stabilises the system.

Let us set $K = 1$ and vary Ω in the neighbourhood of $\Omega = 1/2$. In Figure 3.14, series are shown for $\Omega \in \{0.460, 0.464, 0.500, 0.536, 0.540\}$. Apparently, the system has period-2 behavior in the entire interval $\Omega \in [0.464, 0.536]$. This is interesting, since without the nonlinearity, period-2 solutions were only present for $\Omega = 1/2$. To compare, without the nonlinearity $\Omega = 0.464$ would result in a periodic solution with a rotation number $w = \frac{464}{1000}$.

Instead the nonlinearity causes the system to lock into a period-2 solution. This behavior is called *mode locking* or also *phase locking*.

The tendency of a system to modify its frequency due to external impulses occurs in many situations. For example, crickets in South East Asia are known to chirp at a certain frequency and do so over extensive periods of time. Apparently, the crickets can increase or decrease their own natural chirping frequency depending on stimuli from outside to lock their phase with the other crickets.

For more information on this topic, see [Strogatz, 1994].

Normally, phase-locking is the result of two competing timescales: a natural internal timescale (such as the cricket's preferred chirping rate) and an external forcing time scale (the chirping rate of the other crickets).

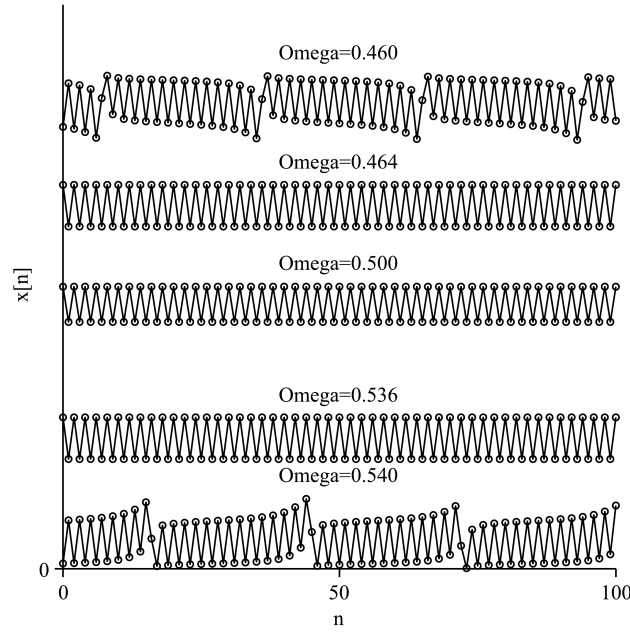


Figure 3.14: phase locking in a period-2 solution for $\Omega \in [0.464; 0.536]$ and $(K = 1)$ in the circle map.

The spontaneous locking into specific modes may seem quite mysterious, but unlocking the secret behind the mode-locking of the circle map will require nothing more than the tools of chapter 2. As we are interested in period-2 solutions $x_{n+2} = x_n = x^*$, it is natural to consider the second iterate $x_{n+2} = f(f(x_n))$ and study it for various values of Ω . Shown in Figure 3.15 are the mappings for $\Omega = 0.464$, 0.500 and 0.536 . As Ω increases, the mappings shift upwards. Around $\Omega = 0.464$ (thin dashed line), the mapping starts intersecting with $y = x$, which lasts up to $\Omega = 0.536$ (dash-dotted line). Thus, the phase-dependent acceleration or deceleration introduced by the nonlinearity, makes that period-2 solutions are possible over a finite range of Ω .

The interval for stable period-2 solutions of $\Omega \in [0.464, 0.536]$ was obtained by trial and error. However, it is possible to calculate the stable regions directly by focusing on the point where period-2 solutions become stable. At this point of bifurcation, in addition to $f(f(x)) = x$, the stability σ is exactly 1 (as can be seen in Figure 3.15; at bifurcation $f(f(x))$ has the same slope as $y = x$). This results in two equations, allowing us to solve for x and Ω in parallel.

The method to solve for the stability region of period-2 solutions with `maple` is shown below. Unfortunately, because of the particular nonlinearities of the problem, it is not possible to use `solve`. However, `fsolve` deals with this problem efficiently, although we will have to take comfort with numerical approximations.

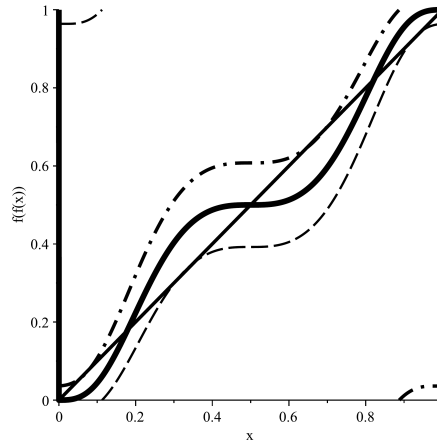


Figure 3.15: The second iterate $f(f(x))$ for the circle map at $K = 1$. Shown are the mappings for $\Omega = 0.464$ (thin dashed line), $\Omega = 0.500$ (solid line) and $\Omega = 0.536$ (dash-dotted line).



Maple

```
Omega := 'Omega':
f2 := x -> f(f(x));
f2mod1 := x -> mod1(f2(x));
df2 := D(f2);
sys := {df2(x) = 1, f2mod1(x) = x}:
fp1 := fsolve(sys, {x=0..1, Omega=0..1}):
fp2 := fsolve(sys, {x=0..1, Omega=0..1}, avoid={fp1}):
fp3 := fsolve(sys, {x=0..1, Omega=0..1}, avoid={fp1, fp2}):
fp4 := fsolve(sys, {x=0..1, Omega=0..1}, avoid={fp1, fp2, fp3}):
sols := fp1, fp2, fp3, fp4;
```

```
sols := {Omega = 0.46302, x = 0.32904}, {Omega = 0.53698, x = 0.06801},
        {Omega = 0.15915, x = 0.75000}, {Omega = 0.84085, x = 0.25000}.
```

The first two solutions are the ones we are looking for; judging from the fact that $x = 0.25$ and $x = 0.75$ are on the line $y = x$ they must correspond to period-1 solutions. Therefore, the stability region for the period-2 solution is $\Omega \in [0.46301; 0.53698]$.

The circle map (3.28) has some very special characteristics for $K = 1$ and $\Omega \in [0, 1]$. To demonstrate this, we introduce a generalized rotation number W defined by

$$W = \frac{f^{(n)}(x_0) - x_0}{n} = \frac{x_n - x_0}{n}. \quad (3.29)$$

It is stressed that $f(x)$ may not contain the modulo operator, as it eliminates information about the number of cycles the system has undergone.

As opposed to w , the generalized rotation number W is by definition a real number. When the solution is periodic, W converges to w as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} W = w. \quad (3.30)$$

This can be shown as follows. Assume that the system has a period- m solution with rotation number w . Now note that

$$W = \frac{x_n - x_0}{n} = \frac{x_m - x_0}{n} + \frac{x_{2m} - x_m}{n} + \dots + \frac{x_n - x_{rm}}{n},$$

where r is the number of full cycles of size m in n . Due to the periodicity, $x_m - x_0 = x_{2m} - x_m = \dots = mw$, by the definition of the rotation number (3.27). Thus,

$$W = \frac{rmw}{n} + \frac{x_n - x_{rm}}{n}.$$

Approximating the number of full cycles r by $r \approx n/m$ and the residual $x_n - x_{rm} \approx mw$ (which is assumed to include any contributions due to transients as well), we obtain that

$$\lim_{n \rightarrow \infty} W \approx \lim_{n \rightarrow \infty} \left(1 + \frac{m}{n}\right) w = w$$

which demonstrates (3.30).

Using (3.29), we can create a plot of W vs. Ω which gives information about the mode-locking behavior of the circle map at $K = 1$. The following **maple** fragment creates this diagram.



Maple

```
restart; with(plots):
f := (x, Omega) -> x + Omega + K/twopi * sin(twopi*x);
twopi := evalf(2*Pi);
K := 1;
N := 200;
x0 := 0;
M := 500; Omin := 0; Omax := 1;
for m from 0 to M do
    Om[m] := evalf(Omin + m*(Omax-Omin)/M);
    xN := x0; for n from 0 to N-1 do xN := evalf( f(xN, Om[m]) ); end do;
    W[m] := (xN-x0)/N;
end do;
plot([seq([Om[m], W[m]], m=0..M)], x, x=0..1, 0..1,
      style=[point, line], symbolsize=1);
```

Note that instead of taking the limit of $n \rightarrow \infty$, W is based on a series of 200 iterations only. Therefore, residuals will not be negligible for periodic solutions with $m > 10$. The result is shown in Figure 3.16. The horizontal plateaus represent the mode-locked states. Clearly, there are many mode-locked states in the system, such as $w = 1/3$, $w = 2/3$, $w = 1/4$, $w = 3/4$ etc. The width of the plateaus decreases as the denominator of w becomes larger. It can be shown that for $K = 1$, the circle map is in a mode-locked state almost everywhere [Schuster, 1995] for $\Omega \in [0, 1]$, see Figure 3.16. In fact this graph forms a so-called *Devil's staircase*, because it consists of an infinite number of plateaus and an infinite number of steps. If one were to climb this staircase step by step, it would take forever to reach the top.

In the case of the circle map at $K = 1$, the set of measure 0 is provided by the quasi-periodic states. Figure 3.17 shows the bifurcation diagram for Ω at $K = 1$ and the corresponding Lyapunov exponents Λ . As can be seen, Λ is generally negative, indicating the periodic states. Upon moving from one mode-locked state to the next, Λ increases to approximately zero, after which it becomes negative again. Interestingly, this situation is entirely opposite from the situation when $K = 0$.

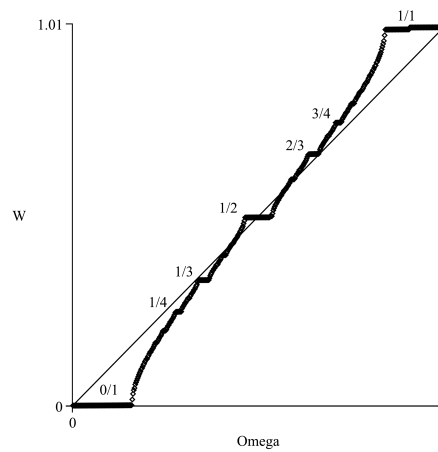


Figure 3.16: Generalized rotation number W as a function of Ω . This is a Devil's staircase.

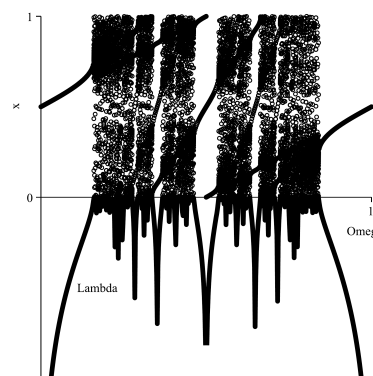


Figure 3.17: bifurcation diagram and Lyapunov exponents

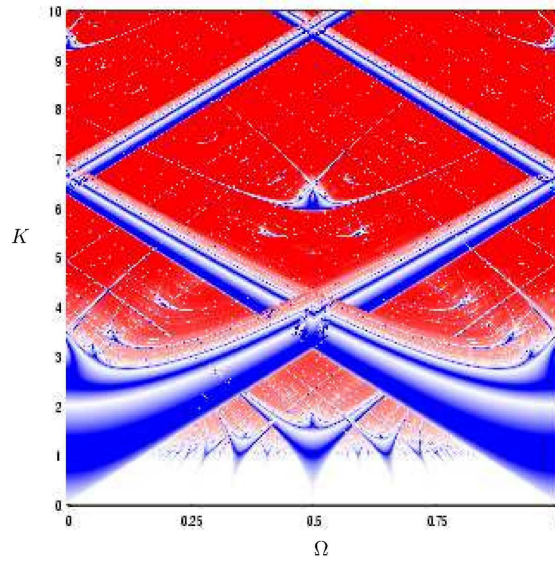


Figure 3.18: Lyapunov exponents Λ in the Ω - K parameter-space. Coloring is $[-1, 1]$; colorscheme: blue=-1; white=0; red=1

In that case, quasi-periodic states are present almost everywhere except where Ω is a rational number.

One may wonder why there are no chaotic solutions, as Λ never seems to get above zero. This is because (3.28) is invertible for $K \leq 1$.

Chaos sets in above $K = 1$, although regions with mode-locking and with chaos are intertwined as a function of K and Ω . Mode-locked states can exist very near to chaotic states in the $\Omega - K$ phase space. Shown in Figure 3.18 are the Lyapunov exponents for a large portion of the Ω - K phasespace. The color code is such that the periodic states for which $\Lambda < 0$ are colored blue, and the chaotic regions for which $\Lambda > 0$ are colored red. For $\Lambda = 0$, corresponding to quasi-periodic states, the color is white.

3.4.6 The standard map

Another widely studied chaotic discrete map is the *standard map*. This system represents the frictionless kicked rotor system (section 3.4.1), but also describes many other systems in the fields of mechanics of particles, accelerator physics, plasma physics, and solid state physics. The standard map is given by

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \quad \text{mod } 1, \\ y_{n+1} &= y_n + \frac{K}{2\pi} \sin(2\pi x_n) \quad \text{mod } 1. \end{aligned} \tag{3.31}$$

Here, x and y are associated with the phase angle and angular momentum, respectively. Clearly, for $K = 0$, the map reduces to the linear map we studied in section 3.4.4.

Unlike the maps we studied so far, the behavior of the standard map depends strongly on its initial conditions. Indeed, chaotic behavior was normally obtained by changing a system parameter, such as r in the case of the logistic map. For the standard map, (quasi-) periodic solutions can coexist with chaotic solutions at fixed K . Figure 3.19 shows a time-series with initial conditions $(x_0, y_0) = (0.4, 0.1)$ at $K = 2$. Although there seems to be some order in the system judging from

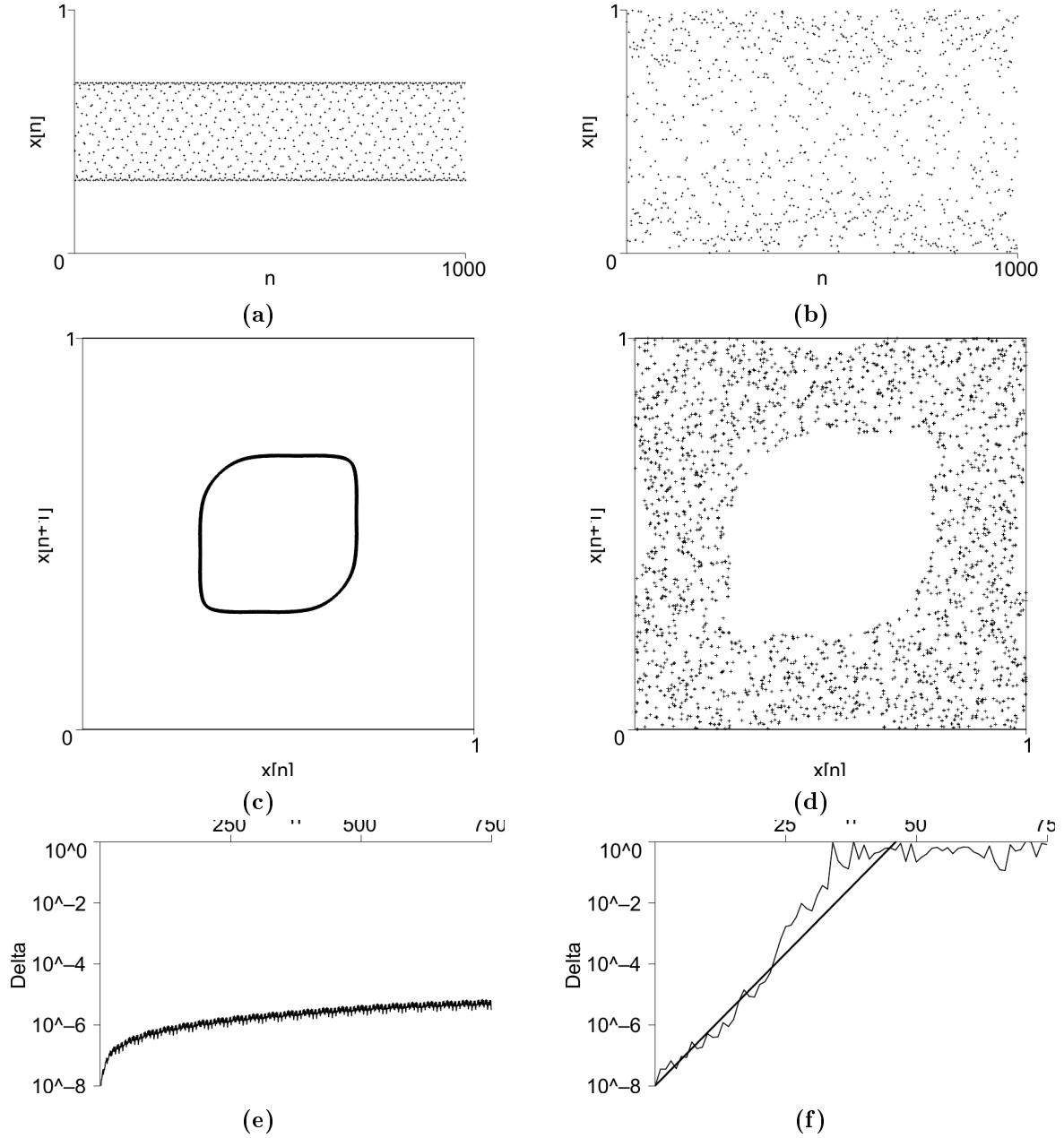


Figure 3.19: Coexistence of chaotic and quasi-periodic solutions at $K = 2$. Note the factor 10 difference in scale for the x -axis between Figure 3.19 and Figure 3.19. (a) Quasi-periodic; series. (b) Chaotic; series. (c) Quasi-periodic; return plot. (d) Chaos; return plot. (e) Quasi-periodic; deviations. (f) Chaos; deviations.

the hexagonal-like patterns, the series seems to traverse an entire interval of x . The return plot of (Figure 3.19) shows that all points fall on one curve, which suggests that the series is either quasi-periodic or chaotic. One can distinguish between the two by studying the sensitivity to initial conditions, and Figure 3.19 shows that deviations seems to saturate at 10^{-5} . The associated Lyapunov exponent $\Lambda \approx 0$, and we conclude that this series is quasi-periodic.

On the other hand, using initial conditions $(x_0, y_0) = (0.1, 0.1)$, the standard map displays chaotic behavior. The series is shown in Figure 3.19, and its return plot in Figure 3.19. The return plot reveals very little order in the system: based on a current value x_n practically all values can be expected for x_{n+1} , except when x_n is near 0.5. The chaotic nature of the series can be verified by the sensitive dependence on initial conditions (Figure 3.19), with an estimated (effective) Lyapunov exponent of $\Lambda \simeq 0.4$.

The return plot for the chaotic series (Figure 3.19) shows no sign of an attractor. In this respect, the standard map is fundamentally different from other 2D discrete maps such as the Hénon map. This is because the standard map is *area preserving*. If one were to start out with a little square of initial conditions and apply ((3.31)) its area will remain the same. This can be shown by calculating the determinant of the Jacobian:



Maple

```
restart: with(LinearAlgebra): with(VectorCalculus, Jacobian):
twopi := 2 * Pi;
f := (x, y) -> x + g(x, y);
g := (x, y) -> y + K/twopi * sin(twopi * x);
J := Jacobian([f(x, y), g(x, y)], [x, y]);
det := Determinant(J);
```

$$J = \begin{pmatrix} 1 + K \cos(2\pi x) & 1 \\ K \cos(2\pi x) & 1 \end{pmatrix}$$

$$\det = 1$$

Although the Jacobian J has a functional dependence on x , its determinant does not. The mapping preserves its area because $\det J = 1$. Recall that the determinant of a matrix J is related to its eigenvalues by $\det J = \sigma_1 \sigma_2$. In the eigenframe, an infinitesimal square of size $\delta_1 \times \delta_2$ would map to $\sigma_1 \delta_1 \times \sigma_2 \delta_2 = \det J \delta_1 \times \delta_2$. Therefore, if $\det J = 1$, areas are preserved. The preservation of area means that this system cannot settle on an attractor, as this requires a convergence of initial conditions (i.e. $|\det J| < 1$) to a small region of space.

The phase space (x, y) of the standard map for $K = 2$ is shown in Figure 3.20. The different colors in the diagram are series obtained with different initial conditions. A large part of the phase space has chaotic behavior (black dots). However, there are islands of order in this 'sea of chaos' (other colors). In particular, there is a large region around $(x, y) = (1/2, 0)$ with quasi-periodic solutions. Note that the grey-coloured quasi-periodic solution around $(1/2, 1/2)$ actually consists of two regions (the other one around $(0, 1/2)$) between which it alternates.

The standard map is periodic (modulo 1) in both x and y . Therefore, the phase space is topologically a *torus*. Imagine that the phase space of Figure 3.20 is glued on a thin sheet of rubber. As $x = 0$ and $x = 1$ represent the same point, we bring them together, obtaining a cylinder. Now, $y = 0$ and $y = 1$ also represent identical points, and on bringing together the two ends of the cylinder, one ends up with a torus (Figure 3.20).

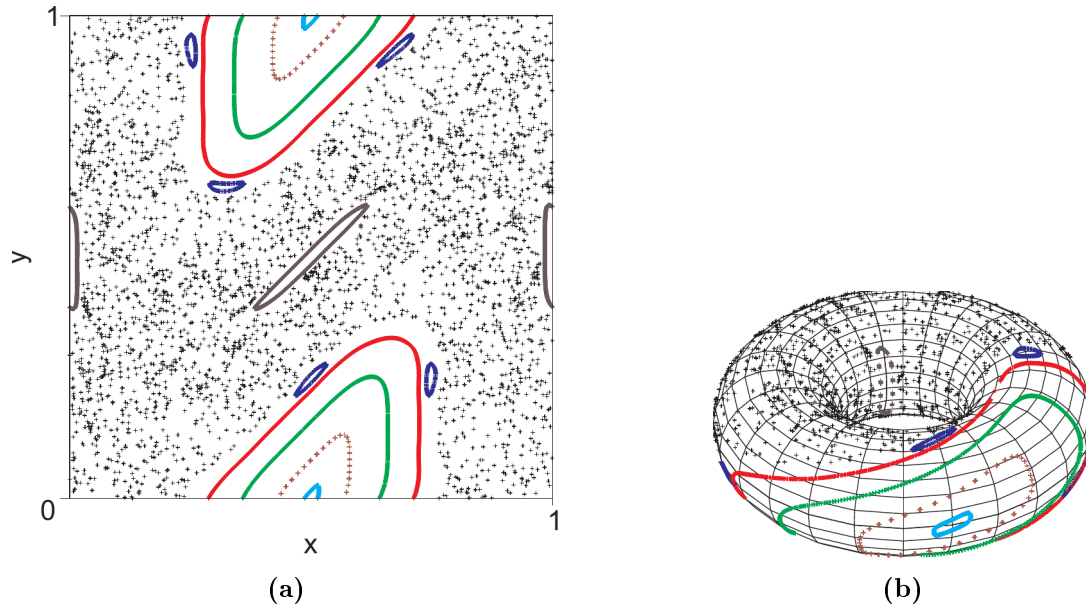


Figure 3.20: Islands of order in a sea of chaos for the standard map at $K = 2$ (a) Phasespace. (b) Mapped onto torus.

3.4.7 Stable and unstable manifolds

The standard map ((3.31)) has a fixed point at $(0,0)$ and the eigenvalues of the Jacobian in the fixed point $J^* = J(0,0)$ are given by

$$\sigma_{1,2} = 1 + \frac{1}{2}K \pm \frac{1}{2}\sqrt{K(K+4)}.$$

For all $K > 0$, $\sigma_1 > 1$ and $0 < \sigma_2 < 1$. Therefore, the fixed point $(0,0)$ is a saddle node. In many ways, this saddle node in the domain is the source of the complex behavior, and we will explore this below.

Let us follow the evolution of a large cloud of initial conditions focused very near the saddle node $(0,0)$.

The initial shape is a small square square, which is shown in Figure 3.21. As before, we take $K = 2$, for which the eigenvalues and eigenvectors can be calculated by

 **Maple**

```
K := 2;
fp := {x=0, y=0};
(v,e) := Eigenvectors(subs(fp, J));
```

Thus, the largest eigenvalue $\sigma_1 = 3.73$ with eigenvector $\mathbf{e}_1 = (0.8, 0.6)$ and the other eigenvalue $\sigma_2 = 0.27$ with eigenvector $\mathbf{e}_2 = (-0.35, 0.97)$. Based on $\mathbf{e}_1$, we expect that the cloud will expand at an angle of about 40° with the horizontal.

Indeed, when we apply ((3.31)) on the square, it deforms into a parallelogram; expanding in the direction of σ_1 and contracting in the direction of σ_2 whilst preserving its area. In the first few iterations, changes are too small to be visible at the size of the domain, but at $n = 3$ (Figure 3.21) the points have escaped the vicinity of the fixed point. Note that due to constant expansion in

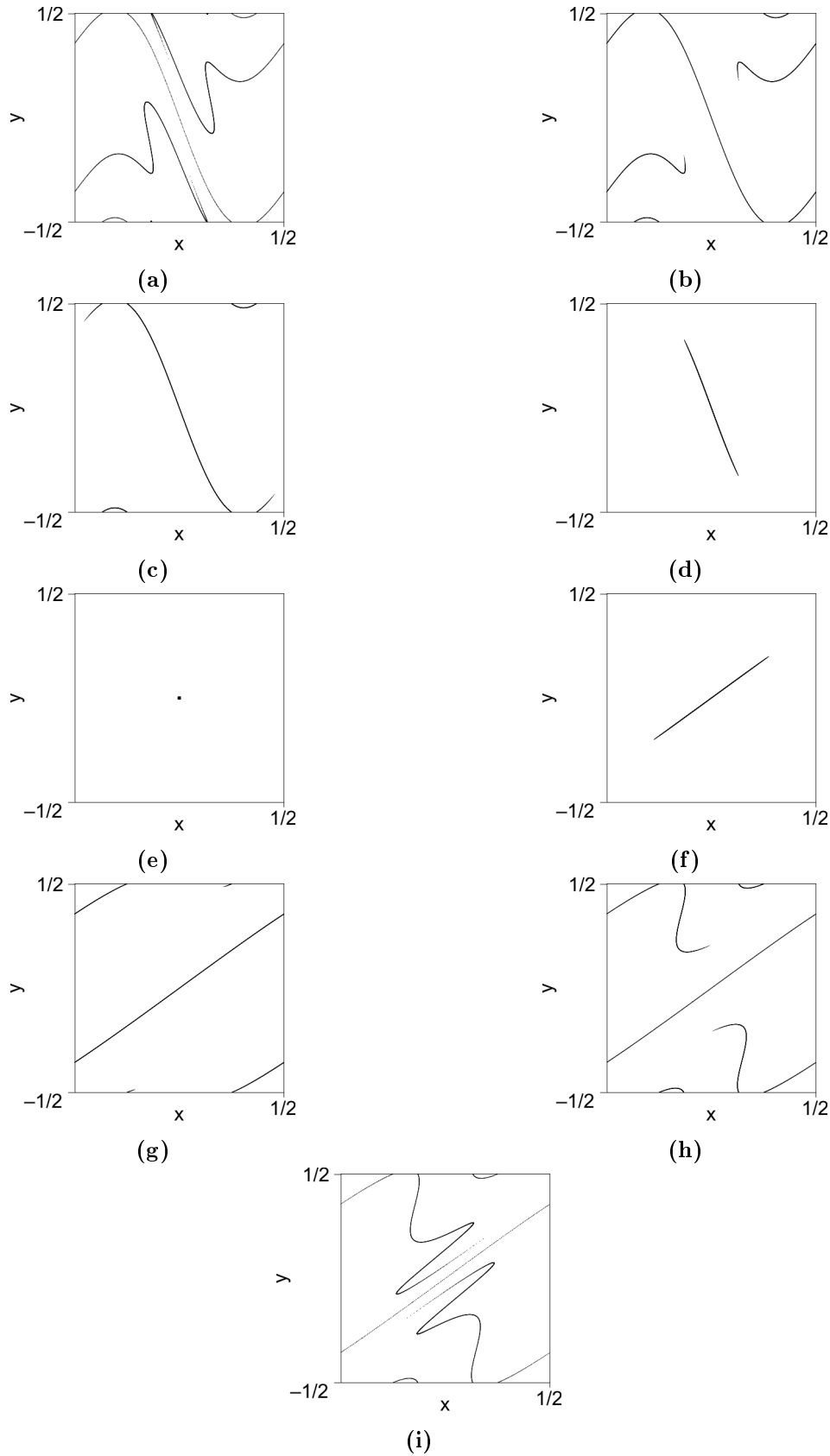


Figure 3.21: Unfolding of the stable and unstable manifolds by following a small cloud of initial conditions around $(0,0)$ (fig e). (f-i) shows the unfolding of the unstable manifold. By iterating the inverse ((3.32)), (a-d) show the unfolding of the stable manifold. (a) $N=-6$. (b) $N=-5$. (c) $N=-4$. (d) $N=-3$. (e) $N=0$. (f) $N=3$. (g) $N=4$. (h) $N=5$. (i) $N=6$.

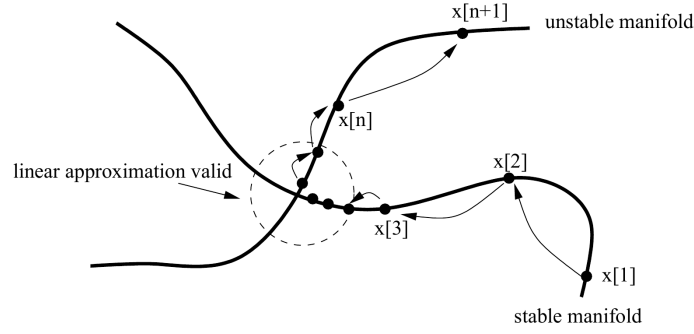


Figure 3.22: Sketch of the stable and unstable manifolds

one direction and contraction into the other, the initial square deforms into a curve. In the next iteration (Figure 3.21), the curve spans the entire domain and at $n = 5$ (Figure 3.21), the curve deflects back towards its origin but now from the stable direction. As the curve approaches the fixed point (Figure 3.21) it starts oscillating back and forth with increasing amplitude as the fixed point approaches.

The curve that is emerging in Figure 3.21 is called the *unstable manifold*, and is denoted by W^u . The unstable manifold W^u is defined as the set of points in the domain which originate from the fixed point. Denoting the map by $\mathbf{f}$, its inverse by $\mathbf{f}^{-1}$ and the fixed point by $\mathbf{x}^*$, the unstable manifold W^u is the set of all $\mathbf{x}$ for which

$$\lim_{n \rightarrow \infty} \mathbf{f}^{(-n)}(\mathbf{x}) = \mathbf{x}^*$$

An example is sketched in Figure 3.22 with a series $\mathbf{x}^u$ which travels away from the fixed point over the unstable manifold. If we invert $\mathbf{f}$, stable directions become unstable and vice versa. Furthermore, the series $\mathbf{x}^u$ is reversed by which it ends up in $\mathbf{x}^*$ in the limit.

If there is a unstable manifold, then there must be a *stable manifold* as well. This is indeed the case, and the stable manifold W^s is defined as

$$\lim_{n \rightarrow \infty} \mathbf{f}^{(n)}(\mathbf{x}) = \mathbf{x}^*$$

Thus, the requirement of W^s is that by iterating $\mathbf{f}$ forward to infinity, one must end up in $\mathbf{x}^*$. To find W_s with `maple`, we employ exactly the same strategy as for W^u but now we iterate with the *inverse* mapping. The inverse of ((3.31)) is given by

$$\begin{aligned} x_n &= f^{-1}(x_{n+1}, y_{n+1}) = x_{n+1} - y_{n+1}, \\ y_n &= g^{-1}(x_{n+1}, y_{n+1}) = y_{n+1} - \frac{K}{2\pi} \sin(2\pi x_n). \end{aligned} \quad (3.32)$$

As mentioned before, by inverting ((3.31)), the stable directions near the fixed point become unstable and vice versa. If we repeatedly apply ((3.32)) to the cloud of initial conditions (Figure 3.21), we obtain after 3 iterations Figure 3.21, after 4 iterations Figure 3.21 and so on. As can be seen, the stable manifold has qualitatively the same features as the unstable manifold, in that it turns back on itself and starts oscillating with increasing amplitude as it approaches the fixed point.

By plotting the stable and unstable manifolds W^s and W^u in one graph (Figure 3.23), we see that the stable and unstable manifolds intersect many times. This means that there are points in the phase space which end up in the saddle node at $(0, 0)$ both in the limit of $n \rightarrow -\infty$ or

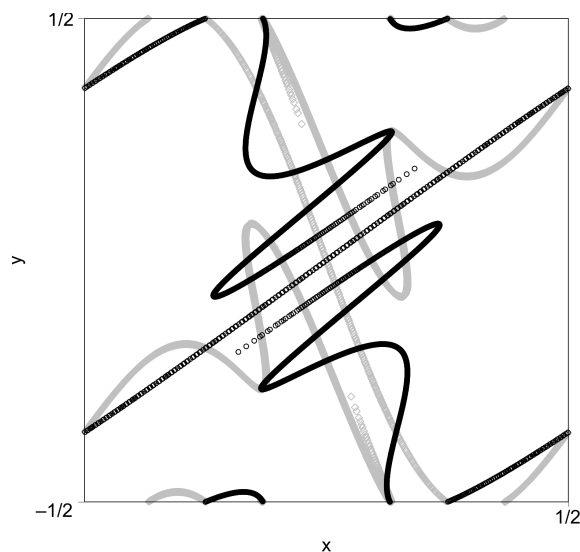


Figure 3.23: Intertwining of the unstable (black circles) and stable (grey diamonds) manifolds, showing how incredibly complex the phase plane is. Note that this only a small part of the manifolds, as we have only iterated 7 times instead of infinite.

$n \rightarrow \infty$. An intersection of a stable and unstable manifold originating from the same fixed point is called a *homoclinic intersection*. If a stable manifold from a fixed point intersects with an unstable manifold from another fixed point, the intersection is called a *heteroclinic* intersection. There are either zero or infinitely many intersections of W^s and W^u . This is easy to verify, because when there is one intersection $\mathbf{x}_i$, applying the mapping $\mathbf{f}$ will generate another intersection. Iterating again will generate another intersection and so on to infinity. The same holds for applying the inverse $\mathbf{f}^{-1}$ on $\mathbf{x}_i$.

For more information, we refer to e.g. [Ott, 1993, Schuster, 1995].

3.5 Exercises

3.5.1 Bouncing Ball

Downloads: BouncingBall.mws, BouncingBallFixedPoint.mws

The simplified and rescaled equations for the bouncing ball on an oscillating plate are

$$\begin{aligned}\phi_{n+1} &= (\phi_n + rv_n) \bmod 2\pi \\ v_{n+1} &= kv_n + (1 + k) \cos(\phi_{n+1})\end{aligned}$$

where v_n represents the ball velocity after the bounce, ϕ_n the phase of the oscillating plate. The parameter k is the collision parameter and is associated with the energy loss at a bounce ($0 < k \leq 1$). The parameter r denotes the non-dimensional angular velocity of the plate.

- Program these equation in **maple**. The modulo-function needs to be user-defined: `modf := (x,y) -> x - floor(x/y)*y`. Don't forget to put an `evalf(...)` in the loop.
- Study the behaviour of this system by looking at the time series (n, v_n) . Look also at plots of (ϕ_n, v_n) and return plots of v_n . Set $k = 0.3, r = 20$. Try also different values of r , and k .
- Take $k = 0.545$ and $r = 3.35$. Describe the behaviour of the system.

- d) Take $k = 0.3$. Make a bifurcation diagram $v_n(r)$ for $r \in [1, 10]$. Zoom in on $r \in [3, 3.5]$.
- e) Find a value for r for which a fixed point is stable.
- f) Prove analytically with `maple` that the fixed point is stable

3.5.2 Dripping Faucet Model

Downloads: `DrippingFaucetModel.mws`

Consider the following damped oscillation as a model for an oscillating droplet at (effective) position $y(t)$

$$y(t) = \exp(-\delta t) [A \cos(\omega t) + B \sin(\omega t)]$$

where δ is a damping factor, and ω the angular velocity. Differentiation with respect to t yields an expression for the velocity $v(t)$. After expressing the constants A and B in terms of the initial position and velocity, y^0 and v^0 , one arrives at the following system:

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = A(t) \begin{pmatrix} y^0 \\ v^0 \end{pmatrix} \quad \text{with} \quad A(t) = \frac{e^{-\delta t}}{\omega} \begin{pmatrix} \omega \cos \omega t + \delta \sin \omega t & \sin \omega t \\ -(\omega^2 + \delta^2) \sin \omega t & \omega \cos \omega t - \delta \sin \omega t \end{pmatrix}$$

Remember that you may copy/paste from a previous exercise.

- a) Program in maple the resulting equations for $y(t, y^0, v^0)$ and $v(t, y^0, v^0)$ as a function of three variables, i.e.

```
y := (t,y0,v0) -> ....
v := (t,y0,v0) -> ....
```

The droplet mass $m(t)$ is given by

$$m(t) = m^0 + \mu t$$

where μ is the flow rate. Each droplet will snap at $t = T$ when $m(T) = m^c$.

The simplified and rescaled equations for a series of droplets $n = 1 \dots N$ read

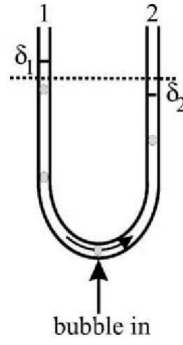
$$\begin{aligned} T_n &= \frac{m^c - m_n^0}{\mu} \\ m_{n+1}^0 &= \frac{1}{2}m^c - \frac{1}{9}y(T_n, y_n^0, v_n^0) \\ y_{n+1}^0 &= 1 \\ v_{n+1}^0 &= v(T_n, y_n^0, v_n^0) \end{aligned} \tag{3.33}$$

Take $m^c = 1$, $\delta = 7.0$, $\omega = 60\pi$ and a flow rate of $\mu = 3$.

- b) Program equations (3.33) in maple and study the time series of T_n for $\mu = 3$. Take $m_1^0 = \frac{1}{2}m^c$, $v_1^0 = 0$, $y_1^0 = 1$ to start the sequence. Take $N = 200$. Show the time series and the return plot.

- c) Set `Digits = 20`. Determine numerically an effective Lyapunov exponent Λ for the system at these settings. To this end create a second series with $\tilde{v}_1^0 = v_1^0 + \epsilon$ with $\epsilon = 10^{-9}$ and find an estimate Λ from the differences between $\tilde{v}$ and v .

- d) What value of Λ do you find when instead you base it on the



e) Set $\mu = 1$. Create a data set and show that a stationary situation is reached. Show that the system can be reduced to the second-order system

$$\begin{aligned} T_{n+1} &= f(T_n, v_n^0) \\ v_{n+1}^0 &= g(T_n, v_n^0) \end{aligned}$$

and determine f and g . Use this to show that the fixed point is indeed stable.

3.5.3 Kicked U-tube

Downloads: `KickedBubble.mws`, `KickedBubble_Phase.mws`

Consider a U-tube filled with water in which air bubbles are injected at regular time intervals at the bottom. The water level relative to the equilibrium level in the right leg is denoted by $\delta = h_2 - h_{eq}$. When the water is brought out of equilibrium, the system will respond by a damped oscillation which is governed by

$$\ddot{\delta} + 2\mu\dot{\delta} + \delta = 0,$$

with $\mu \in [0, 1]$ a frictional parameter.

a) Verify that the solution to this equation is given by

$$\begin{aligned} \delta(t) &= A(t)\delta_0 + B(t)\dot{\delta}_0, \\ \dot{\delta}(t) &= -B(t)\delta_0 + C(t)\dot{\delta}_0, \end{aligned}$$

with $\delta_0, \dot{\delta}_0$ the initial conditions and coefficients A, B, C given by

$$\begin{aligned} A &= e^{-\mu t} \frac{\mu \sin(\omega t) + \omega \cos(\omega t)}{\omega} \\ B &= e^{-\mu t} \frac{\sin(\omega t)}{\omega} \\ C &= -e^{-\mu t} \frac{\mu \sin(\omega t) - \omega \cos(\omega t)}{\omega}. \end{aligned}$$

Here $\omega = \sqrt{1 - \mu^2}$.

b) The bubbles are injected at regular time intervals τ . A bubble entering the tube chooses one of the sides immediately, depending on the direction of the flow at that moment. Therefore, the water level changes instantly in one of the legs and the water velocity is unchanged. This can be mathematically represented by

$$\dot{\delta}_{new} = \dot{\delta}_{old}, \quad \delta_{new} = \delta_{old} - K \operatorname{sgn}(\dot{\delta}_{old})$$

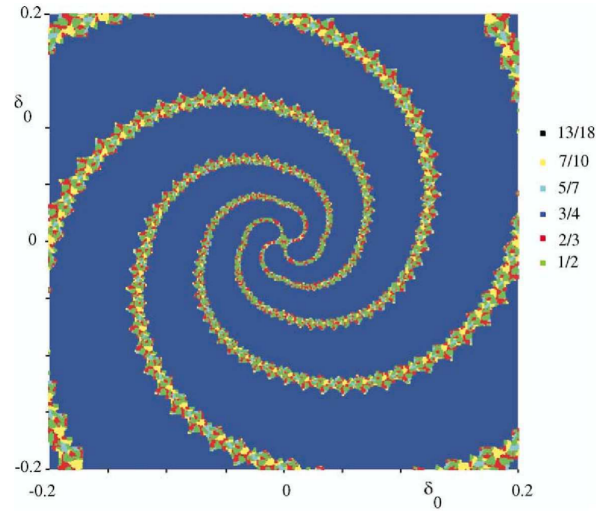


Figure 3.24: Dependence of solution on initial conditions for $K = 0.1$, $\mu = 4.17 \times 10^{-3}$ and $\phi = 0.76$.

Show that the dynamics of the U-tube with the bubbles are governed by the following discrete mapping:

$$\begin{aligned}\delta_{n+1} &= A(\tau)\delta_n + B(\tau)\dot{\delta}_n - K\text{sgn}(\dot{\delta}_{n+1}) \\ \dot{\delta}_{n+1} &= -B(\tau)\delta_n + C(\tau)\dot{\delta}_n\end{aligned}$$

Use the Maple command `signum` for the `sgn` operator, and set `_Envsignum0 := 1` to change the default behavior of `signum(0)=0` to `signum(0)=1`.

c) Take $\mu = 0.01$, $K = 0.1$ and iterate at least 1000 steps to remove transients. Plot $(\delta_n, \dot{\delta}_n)$ for various $\phi \in [0, 1]$. Here ϕ is the ratio of bubble injection time and the oscillation time: $\phi = T_k/T_o = \tau/2\pi$.

d) Create a bifurcation diagram at $K = 0.1$ and $\mu = 0.01$ for $\phi \in [0, 1]$. Is this an example of Feigenbaum's universality?

e) Take $\mu = 0.01$, $K = 0.1$ and $\phi = 0.76$ and vary the initial conditions. Can this sensitive dependence on initial conditions be called chaos?

f) Use the file `KickedBubble_Phase.mws` to study the behavior of the system in the phase space. For $K = 0.1$, $\mu = 0.01$ and $\phi = 0.76$, calculate the rotation number $\#revolutions/\#kicks$ for various initial conditions.

g) Take $K = 0.1$ and calculate the period-1 solution. Note that there are two possibilities due to the `sgn` operator. Plot the locations in the plane for $\mu = 0.01$, $\phi = 0.76$ and verify numerically that this is indeed a fixed point.

h) Calculate the stability of the period-1 solutions and show (graphically or mathematically) that this solution is stable for all μ and τ .

4 Continuous Systems

4.1 Fixed points and stability

A first-order continuous autonomous system is given by

$$\dot{x} = f(x) \quad (4.1)$$

where $\dot{x} = dx/dt$. An example is

$$\dot{x} = x(1 - x) \quad (4.2)$$

The word autonomous means that the right-hand of (4.1) has no explicit time dependence; the dependence on time is implicit through the variable x , i.e. $f(x(t))$. For non-autonomous systems the explicit time-dependence is indicated separately

$$\dot{x} = f(x; t) \quad (4.3)$$

An example of a non-autonomous system is

$$\dot{x} = x(1 - x) + \sin(t) \quad (4.4)$$

Below we will deal with autonomous systems only.

An analytical solution to (4.1) may be sought by the following approach

$$\frac{dx}{dt} = f(x) \quad \rightarrow \quad \frac{dx}{f(x)} = dt \quad \rightarrow \quad \int \frac{1}{f(x)} dx = \int dt = t + c \quad (4.5)$$

where c is an integration constant which will be determined by the initial condition $x(0)$. The success of the method depends on whether one can find an analytical solution to the integral over x . As an example, consider

$$\frac{dx}{dt} = x - x^3 \quad (4.6)$$

Following (4.5) we are to solve

$$\int \frac{dx}{x - x^3} = t + c \quad (4.7)$$

One can rewrite the fraction in three separate terms

$$\frac{1}{x - x^3} = \frac{1}{x(1 - x)(1 + x)} = \frac{A}{x} + \frac{B}{1 - x} + \frac{C}{1 + x}$$

where the constants A, B, C can be found by elaborating the expression at the right-hand side

$$\frac{A(1-x^2) + Bx(1+x) + Cx(1-x)}{x-x^3} = \frac{1}{x-x^3}$$

Working out the numerator and collecting terms with x, x^2 and no x separately, one obtains three equations from which A, B, C can be determined

$$A = 1 \quad B + C = 0 \quad -A + B - C = 0$$

hence $A = 1, B = 1/2, C = -1/2$ and one can proceed to integrate (4.7):

$$t + c = \int \frac{dx}{x-x^3} = \frac{1}{2} \int \left[\frac{2}{x} + \frac{1}{1-x} - \frac{1}{1+x} \right] dx$$

$$2(t+c) = \log x^2 - \log |1-x^2| = \log \left[\frac{x^2}{|1-x^2|} \right]$$

Inverting the last terms and using the initial condition $x(0) = x_0$, one gets an expression for x directly in terms of t and x_0

$$x(t) = \frac{x_0}{\sqrt{x_0^2 + (1-x_0^2)e^{-2t}}} \quad (4.8)$$

The following commands make use of maple's ability to find analytical solutions:

 **Maple**

```
restart: with(plots):
f := x->x-x^3;           # define f(x)
eq:=diff(x(t),t)=f(x(t)); # define differential eq
sol:=dsolve({eq,x(0)=x0},x(t)); # solve it analytically
x0 := 0.1;               # set initial condition
plot(rhs(sol),t=0..6);   # plot the solution
```

Yet in many cases you (or maple) will not be so fortunate to find an analytical solution for $x(t)$. This is either because one cannot evaluate the integral $\int [f(x)]^{-1} dx$, or because one cannot invert the result which is required to get x expressed as a function of t .

An example, relevant to physics, is

$$\dot{x} = -x + \tanh\left(\frac{x}{T}\right)$$

where T is a parameter. This equation describes the evolution of the magnetization of a ferromagnetic system at temperature T . The problem here is that one cannot find an expression for the integral $\int [-x + \tanh(x/T)]^{-1} dx$. We will deal with this system later in more detail.

Fortunately it is still possible to get a good understanding of the behaviour of the system even in absence of closed form analytical solutions. Let us revisit example (4.6). A plot of $f(x) = x - x^3$ is shown in Figure 4.1. What happens with the evolution of $x(t)$ when you start at $x(0) = 1.5$? The rate of change, $\dot{x}$, at this point is given by $f(1.5)$. Looking at the graph in Figure 4.1 one sees that $f(1.5) < 0$, hence the rate of change is negative, which means that x will decrease (as indicated by the arrow on the x -axis). While x decreases, the rate of change $\dot{x} = f(x)$ still remains negative, so $x(t)$ continues to decrease until $x = 1$ has been reached. At this point the rate of change has

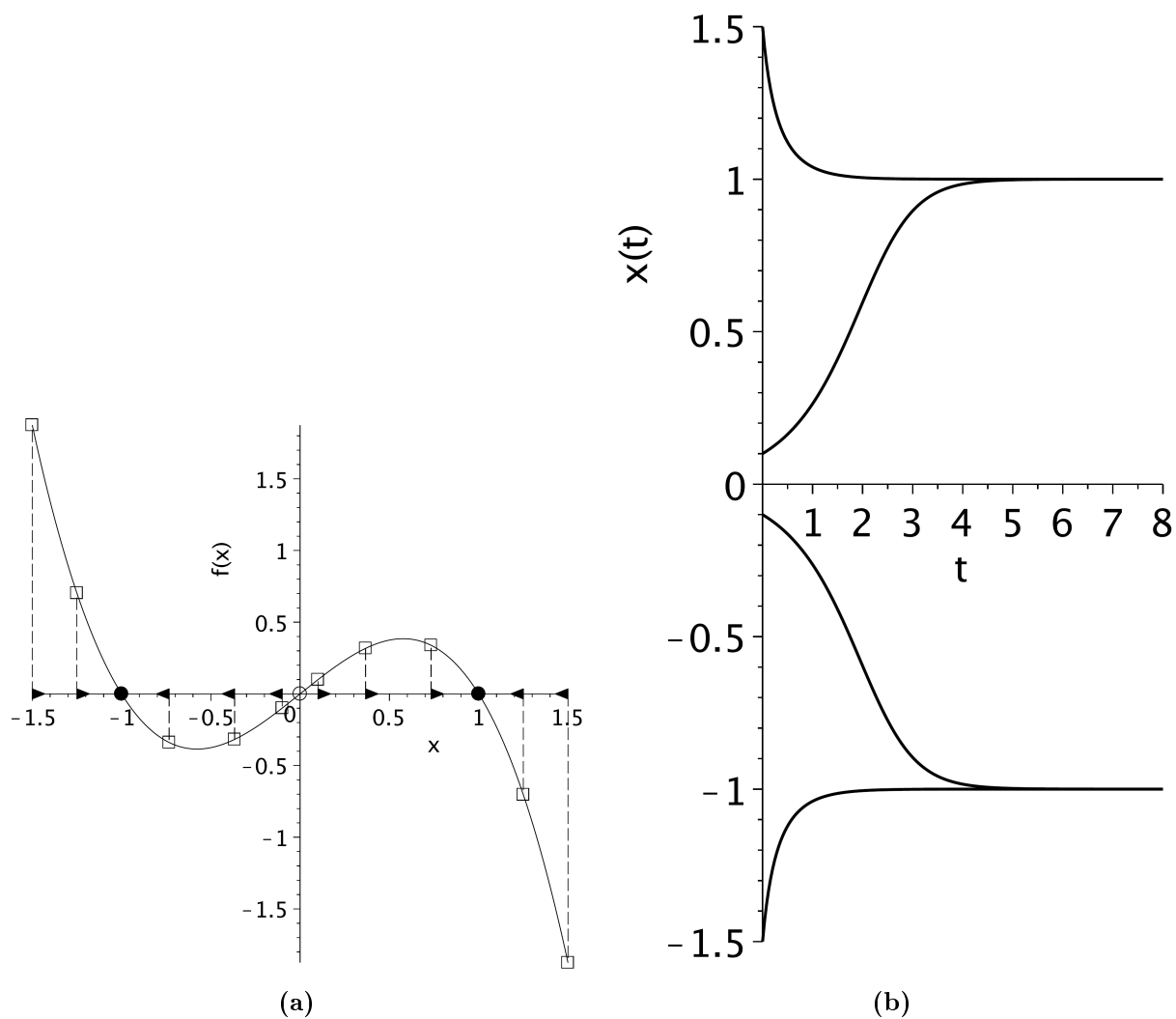


Figure 4.1: Graphs for the system $\dot{x} = f(x) = x - x^3$. (a) Graph of $f(x) = x - x^3$. (b) Evolution of x from four initial conditions.

become zero, $\dot{x} = f(1) = 0$, so x will no longer change. If on the other hand one starts at, say, $x_0 = 0.1$, one finds $f(0.1) > 0$ implying that x will increase (indicated by the arrow pointing to the right). Since $f(x)$ remains positive, x increases until it again arrives at $x = 1$ where the rate of change is zero. Such points where $\dot{x} = 0$ are called *fixed points*.

Fixed points can be readily located by detecting where the graph of $f(x)$ crosses the x -axis. Looking at Figure 4.1 one notices three fixed points $x_* = -1$, $x_* = 0$ and $x_* = 1$, respectively. The fixed point $x_* = 0$ differs from the other two because it is unstable. If one starts exactly at $x(0) = x_* = 0$ then also $x(t) = 0$ for all $t > 0$. But a slight perturbation will cause x to evolve towards one of the stable fixed points $x_* = \pm 1$.

The set of initial conditions that lead to a certain fixed point is called the *domain of attraction* of that fixed point. In the example the attraction domain of $x_* = 1$ is $\langle 0, \infty \rangle$, and that of $x_* = -1$ is $\langle -\infty, 0 \rangle$. The unstable fixed point $x_* = 0$ has no attraction domain by definition.

The graphs in Figure 4.1, which represent the numerical solution to (4.6) for different initial conditions, were made using the following **maple** commands



```
restart: with(plots):
f := x->x-x^3;                               # define f(x)
eq:=diff(x(t),t)=f(x(t));                     # define differential eq
sol:=dsolve({eq,x(0)=0.1},x(t),type=numeric); # solve it numerically
odeplot(sol,[t,x(t)],0..6);                   # plot the solution
```

There was of course no need to employ a numerical procedure since we already had derived the analytical solution (4.8) (which yield the same graphs), but, as mentioned above, having an analytical solution at ones disposal does not happen often, which is why the numerical version is probably more useful in general.

Apart from the numerical/analytical solution, one could have made a sketch of the evolution oneself based solely on the graph of $f(x)$ in Figure 4.1: take for instance the evolution starting at $x_0 = 0.1$. Initially the rate of change is only moderately positive, but as $x(t)$ slowly increases to larger values also $f(x)$ becomes larger, thus speeding up the increase. As x gets closer to $x_* = 1$, the rate of change decreases and the convergence to 1 slows down and ultimately comes to a halt at $x = 1$. In this way one can qualitatively well understand the evolution trajectories of Figure 4.1.

The graphical method outlined above works well for arbitrary systems $\dot{x} = f(x)$. First one makes a graph of $f(x)$ and locates the zero crossings to find the fixed points. If a zero-crossing looks similar to Figure 4.2, in the sense that if the derivative of f with respect to x evaluated in the fixed point is negative, one deals with a stable fixed point. If the derivative is positive $f'(x_*)$, such as depicted in Figure 4.2, the fixed point is unstable. Below we will put this issue on a firmer footing.

4.1.1 Local stability analysis of fixed points

Here we pursue a more mathematically oriented approach to determine the stability of fixed points to $\dot{x} = f(x)$. As said, a fixed point x_* satisfies

$$f(x_*) = 0 \tag{4.9}$$

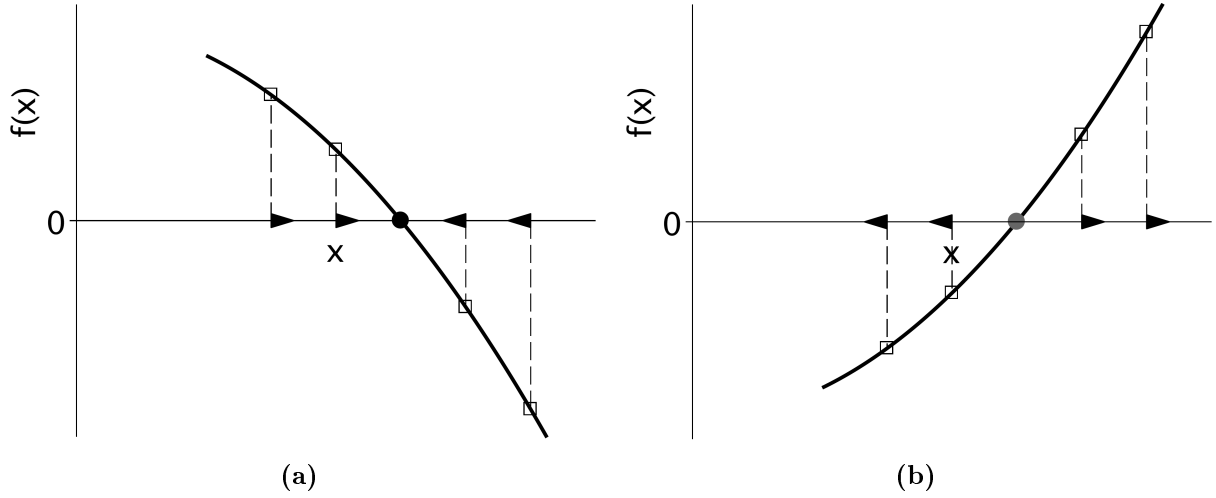


Figure 4.2: Stable and unstable fixed points. (a) Stable fixed point: $f'(x_*) < 0$. (b) Unstable fixed point: $f'(x_*) > 0$.

To investigate the stability of a fixed point, put the system in the fixed point and apply a perturbation $x(0) = x_* + \epsilon(0)$, where $\epsilon(0) \ll 1$. To see whether the system will return to the fixed-point, or whether it will diverge even further from it, one is interested in the evolution of the deviations from the fixed point: $\epsilon(t) = x(t) - x_*$. Differentiation with respect to t followed by a Taylor expansion gives

$$\dot{\epsilon} = \dot{x} = f(x_* + \epsilon) = f(x_*) + \epsilon f'(x_*) + \mathcal{O}(\epsilon^2)$$

with $f' = df/dx$. The first term at the right is zero because of (4.9). Neglecting the higher-order terms in ϵ , one is left with $\dot{\epsilon} = \epsilon f'(x_*)$. Since $\sigma = f'(x_*)$ is constant, the solution for $\epsilon(t)$ is given by

$$\epsilon(t) = \epsilon(0) e^{\sigma t} \quad \text{with} \quad \sigma = f'(x_*)$$

Requiring $\epsilon(t) \rightarrow 0$ yields the condition $\sigma < 0$. Hence we conclude

$$\sigma = f'(x_*) \quad \begin{cases} \sigma > 0 & x_* \text{ is an unstable fixed point} \\ \sigma < 0 & x_* \text{ is a stable fixed point} \end{cases} \quad (4.10)$$

which is consistent with Figure 4.2.

Let us apply this to example (4.6). The derivative is $f'(x) = 1 - 2x^2$, so for $x_* = \pm 1$ one has $\sigma = -1$, demonstrating that both fixed points are stable. For $x_* = 0$, one gets $\sigma = 1$ hence it is unstable.

4.2 Pitchfork bifurcation

A bifurcation is the process where one or more fixed points are created or annihilated due to changes in an external parameter of the system. It turns out that the number of commonly occurring bifurcation *types* is limited. Apparently there is only a limited number of possibilities to create or destroy fixed-points. We treat the three most common ones below.

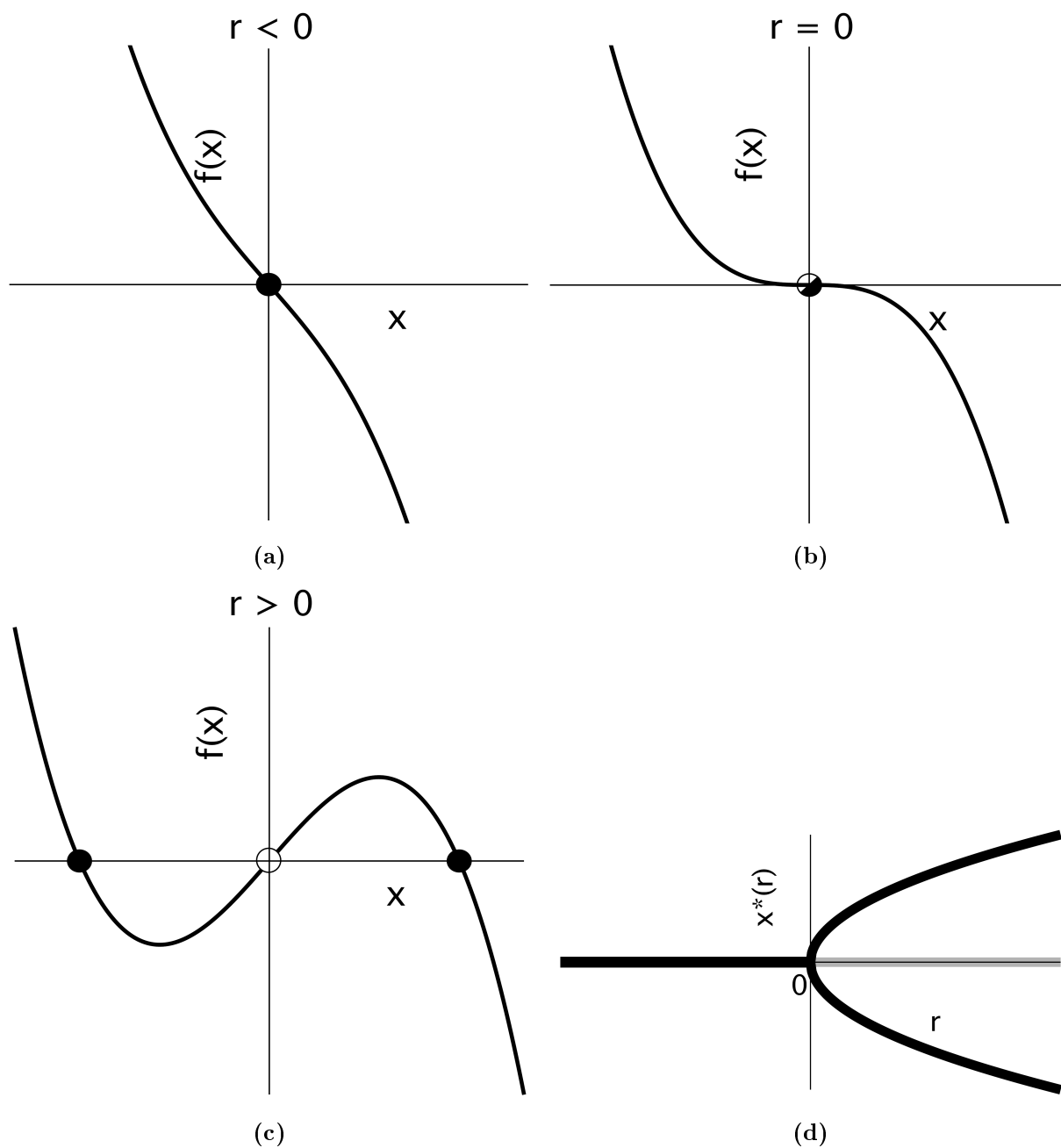


Figure 4.3: Pitchfork bifurcation in $\dot{x} = rx - x^3$. The bifurcation diagram in (d) corresponds to graphs (a)-(c). A black color means that the fixed point branch $x_*(r)$ is stable, whereas gray means unstable. (d) Bifurcation diagram.

As an example, let us add a parameter r to the system (4.6):

$$\dot{x} = f_r(x) = rx - x^3$$

where r can be positive or negative (or zero). Sketches of $f_r(x)$ for various r are given in Figure 4.3. One observes that for $r < 0$ there is a single fixed point $x_* = 0$ which is stable. At $r = 0$ the stability of this fixed point changes and becomes unstable for $r > 0$. However, also two new (stable) fixed points have been created. Solving for $f(x_*) = 0$ gives an expression for the fixed points as a function of r ; the stability is then given by $\sigma = f'_r(x_*) = r - 3x_*^2$:

$$\begin{array}{lll} x_* = 0 & \sigma = r & \text{stable for } r < 0 \\ x_* = \pm\sqrt{r} & \sigma = -2r & \text{stable for } r > 0 \end{array}$$

These three branches have been plotted in Figure 4.3. The $x_*(r) = 0$ branch changes stability around $r = 0$ (indicated by the color change from black to gray). This type of bifurcation is called a *pitchfork* bifurcation – 'bifurc' is the Latin word for fork.

4.3 Saddle-node bifurcation

A so-called *saddle-node* bifurcation is associated with

$$\dot{x} = f_r(x) = r - x^2$$

The corresponding plots for various r are displayed in Figure 4.4. For $r < 0$ there is no fixed point, but the situation changes at $r = 0$, as for $r > 0$ two fixed points have been *created* simultaneously, one stable and one unstable. The fixed points are $x_*(r) = \pm\sqrt{r}$. The stability follows from $\sigma = f'_r(x_*) = -2x_*$:

$$\begin{array}{lll} r > 0 & x_* = \sqrt{r} & \sigma = -2\sqrt{r} \quad \text{stable} \\ r > 0 & x_* = -\sqrt{r} & \sigma = 2\sqrt{r} \quad \text{unstable} \end{array}$$

The branches have been plotted in Figure 4.4. Note the essential differences with the pitchfork bifurcation displayed in Figure 4.3: there are no fixed points for $r < 0$, but when r changes sign *two* fixed points are created 'out of the blue', a stable and an unstable one.

4.4 Transcritical bifurcation

The third type is called *transcritical* bifurcation, associated with a system of the form

$$\dot{x} = f_r(x) = rx - x^2$$

The fixed points follow from $f_r(x_*) = 0$, and the stability from $\sigma = f'_r(x_*) = r - 2x_*$

$$\begin{array}{lll} x_* = 0 & \sigma = r & \text{stable for } r < 0 \\ x_* = r & \sigma = -r & \text{stable for } r > 0 \end{array}$$

The corresponding plots are given in Figure 4.5: one observes that the branches exchange their stability around $r = 0$. This 'exchange of stability' characterizes a transcritical bifurcation.

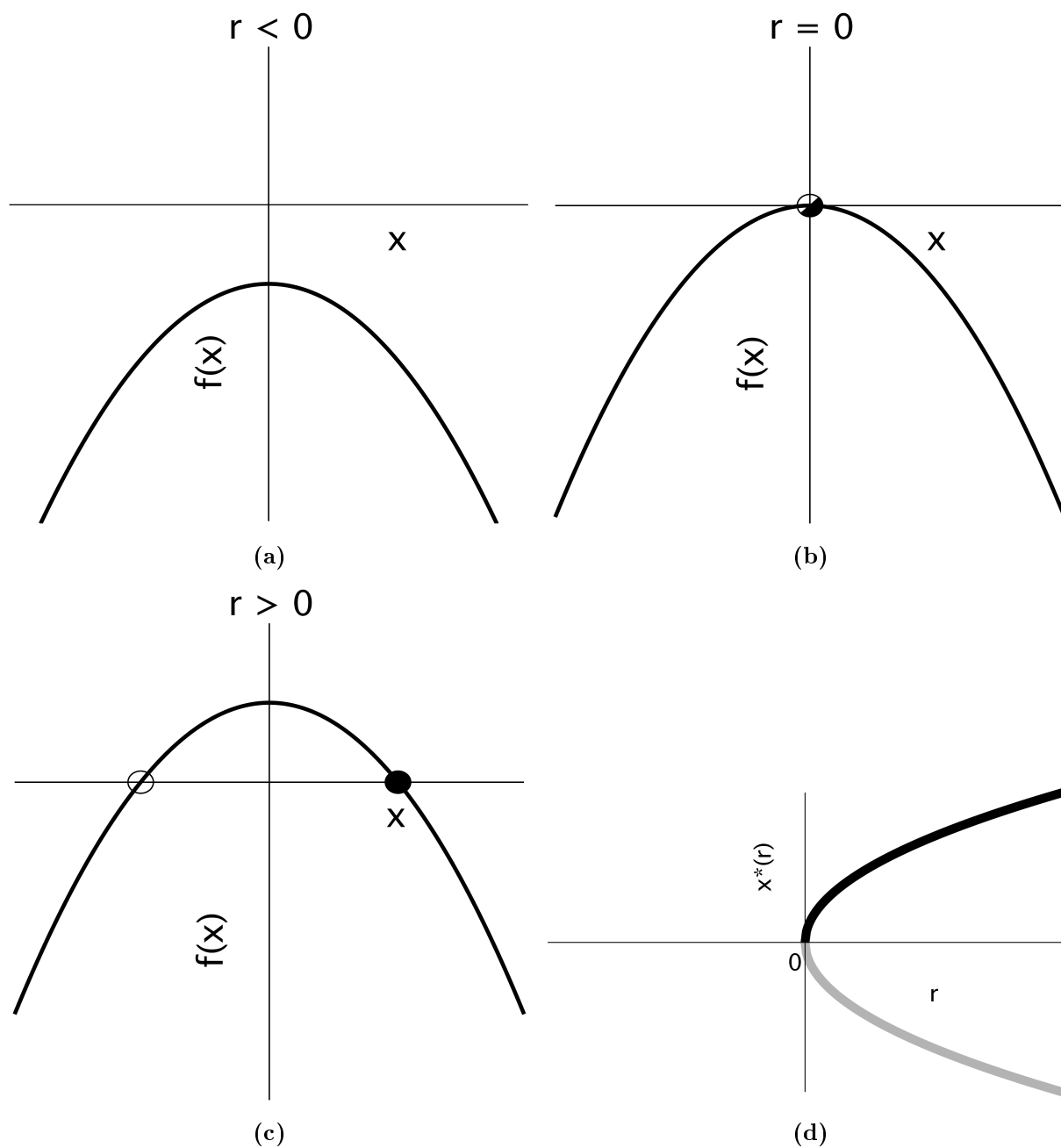


Figure 4.4: Saddle node bifurcation in $\dot{x} = r - x^2$. The bifurcation diagram in (d) corresponds to graphs (a)-(c). A black color means that the fixed point branch $x_*(r)$ is stable, whereas gray means unstable. (d) Bifurcation diagram.

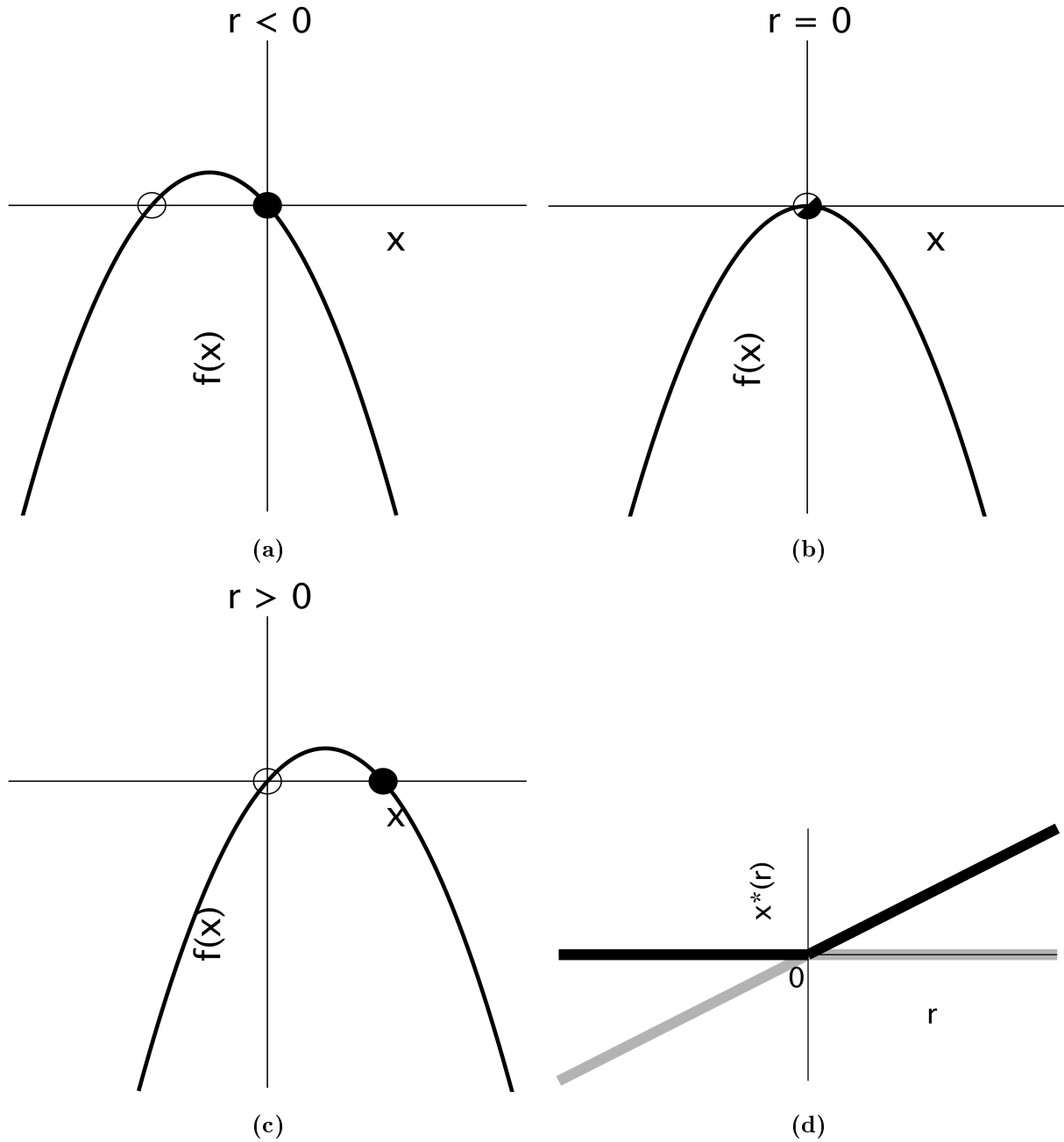


Figure 4.5: Transcritical bifurcation in $\dot{x} = rx - x^2$. The bifurcation diagram in (d) corresponds to graphs (a)-(c). A black color means that the fixed point $x_*(r)$ is stable, whereas gray means unstable. (d) Bifurcation diagram.

4.5 Universality of bifurcations: normal forms

Although the number of one-dimensional non-linear autonomous systems is infinite, the idea is that commonly only a limited number of bifurcation types occur. The underlying reason is that close to the bifurcation point, the system can be well approximated by the first terms of a Taylor series, of which only a few forms are most likely to occur. Consider the generic autonomous system $\dot{x} = f(x, \mu)$ which undergoes a bifurcation at $\mu = \mu_c$. The fixed point(s) around bifurcation $x_*(\mu)$ satisfy $f(x_*, \mu) = 0$. The stability is $\sigma = f'(x_*(\mu), \mu)$. Let $x_c = x_*(\mu_c)$. Precisely at the bifurcation $\sigma = f'(x_c, \mu_c) = 0$. Expanding $f(x, \mu)$ via a Taylor series in x and μ around (x_c, μ_c)

$$\begin{aligned} \dot{x} = f(x, \mu) = & f(x_c, \mu_c) + (x - x_c) \left. \frac{\partial f}{\partial x} \right|_{x_c, \mu_c} + (\mu - \mu_c) \left. \frac{\partial f}{\partial \mu} \right|_{x_c, \mu_c} \\ & + (\mu - \mu_c)(x - x_c) \left. \frac{\partial^2 f}{\partial x \partial \mu} \right|_{x_c, \mu_c} + \frac{1}{2}(x - x_c)^2 \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \mu_c} \\ & + \frac{1}{6}(x - x_c)^3 \left. \frac{\partial^3 f}{\partial x^3} \right|_{x_c, \mu_c} + \dots \end{aligned}$$

The first two terms are zero. Defining $y = x - x_c$ and $\nu = \mu - \mu_c$ gives

$$\dot{y} = a\nu + by\nu + cy^2 + dy^3 + \dots$$

where $a = \left. \frac{\partial f}{\partial \mu} \right|_{x_c, \mu_c}$, $b = \left. \frac{\partial^2 f}{\partial x \partial \mu} \right|_{x_c, \mu_c}$, etc. are constants. If a is non-zero then one can neglect the higher order term $by\nu$ since we look very closely at the bifurcation point, i.e. $|y| \ll 1$ and $|\nu| \ll 1$.

We will distinguish some cases:

- If $a \neq 0$ and $c \neq 0$, one ends up to leading order with $\dot{y} = a\nu + cy^2$. After rescaling the variables y and ν by $z = -cy$, $r = -ac\nu$ this can be converted to $\dot{z} = r - z^2$, i.e. the *normal form* of a saddle-node bifurcation studied above (Figure 4.4).
- If $a = 0$, and $b \neq 0$, $c \neq 0$, one gets $\dot{y} = by\nu + cy^2$, which after rescaling by $z = -cy$, $r = b\nu$ acquires the normal form $\dot{z} = rz - z^2$, i.e. that of a transcritical bifurcation (Figure 4.5).
- If $a = 0$, and $b \neq 0$, but $c = 0$, one must include the cubic term: $\dot{y} = by\nu + dy^3$. After rescaling ($z = \sqrt{|d|}y$, $r = b\nu$) one acquires the normal form $\dot{z} = zr + \text{sign}(d)z^3$. So, if $d < 0$, one retrieves the normal form $\dot{z} = rz - z^3$ of the pitchfork bifurcation studied in Figure 4.3. But note that if $d > 0$, one has $\dot{z} = rz + z^3$. By mere rescaling one cannot map this expression to $\dot{z} = zr - z^3$, so it represents a separate case. This situation is called *subcritical* pitchfork bifurcation. The graphs and corresponding diagram are depicted in Figure 4.6. It bears a strong resemblance with Figure 4.3, but the stability of the respective branches is reversed.

4.6 Example of bifurcations: Ferromagnetic system

The governing equation for the mean magnetization x of a ferromagnetic system at (rescaled) temperature T is given by

$$\dot{x} = f(x, T) = -x + \tanh\left(\frac{x + m}{T}\right) \quad (4.11)$$

where m represents an externally applied magnetic field. More information about this system can be found in the Infobox below. First we consider the case without external magnetic field $m = 0$.

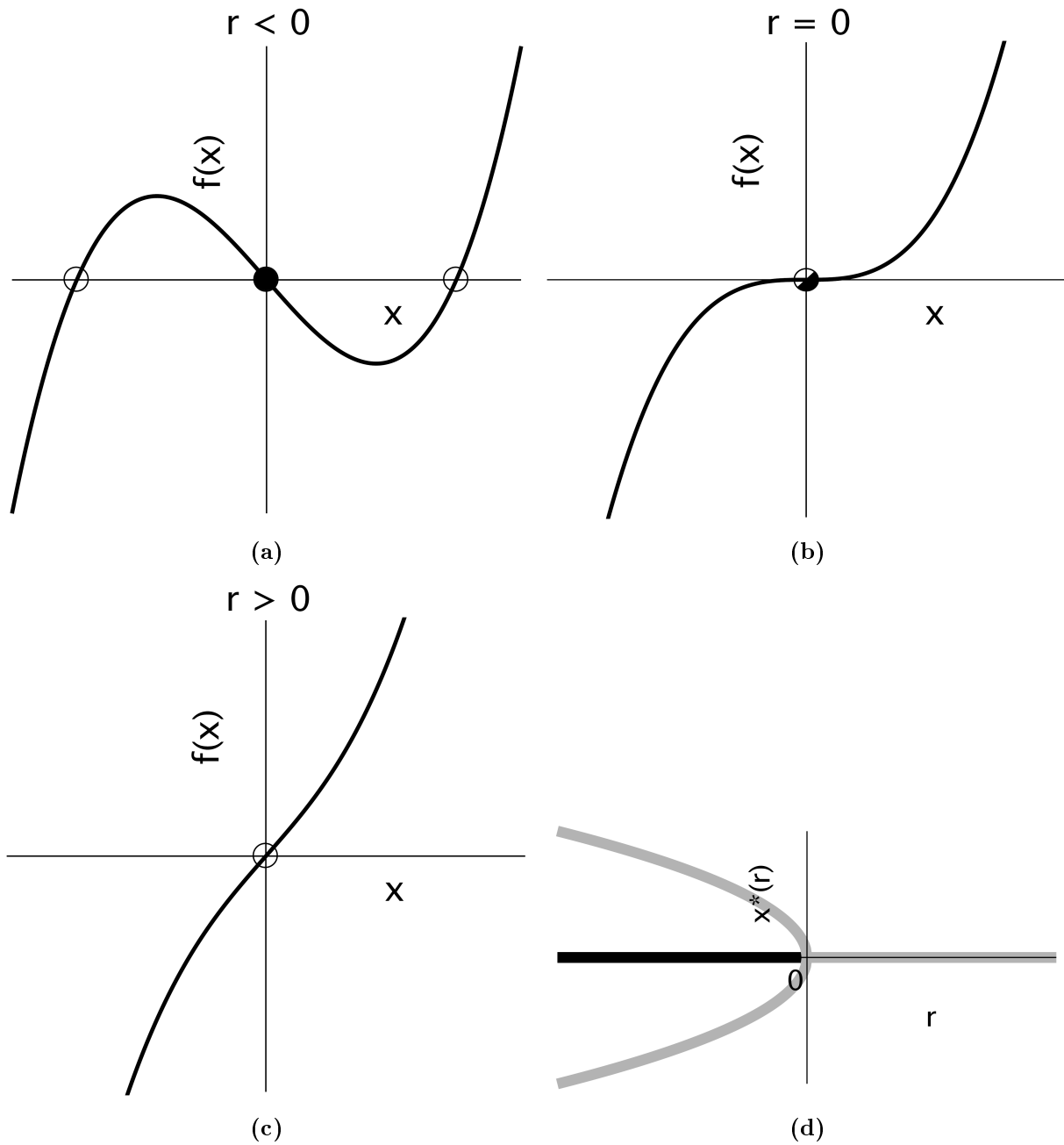


Figure 4.6: Subcritical pitchfork bifurcation in $\dot{x} = rx + x^3$. The bifurcation diagram in (d) corresponds to graphs (a)-(c). A black color means that the fixed point $x_*(r)$ is stable, whereas gray means unstable. (d) Bifurcation diagram.

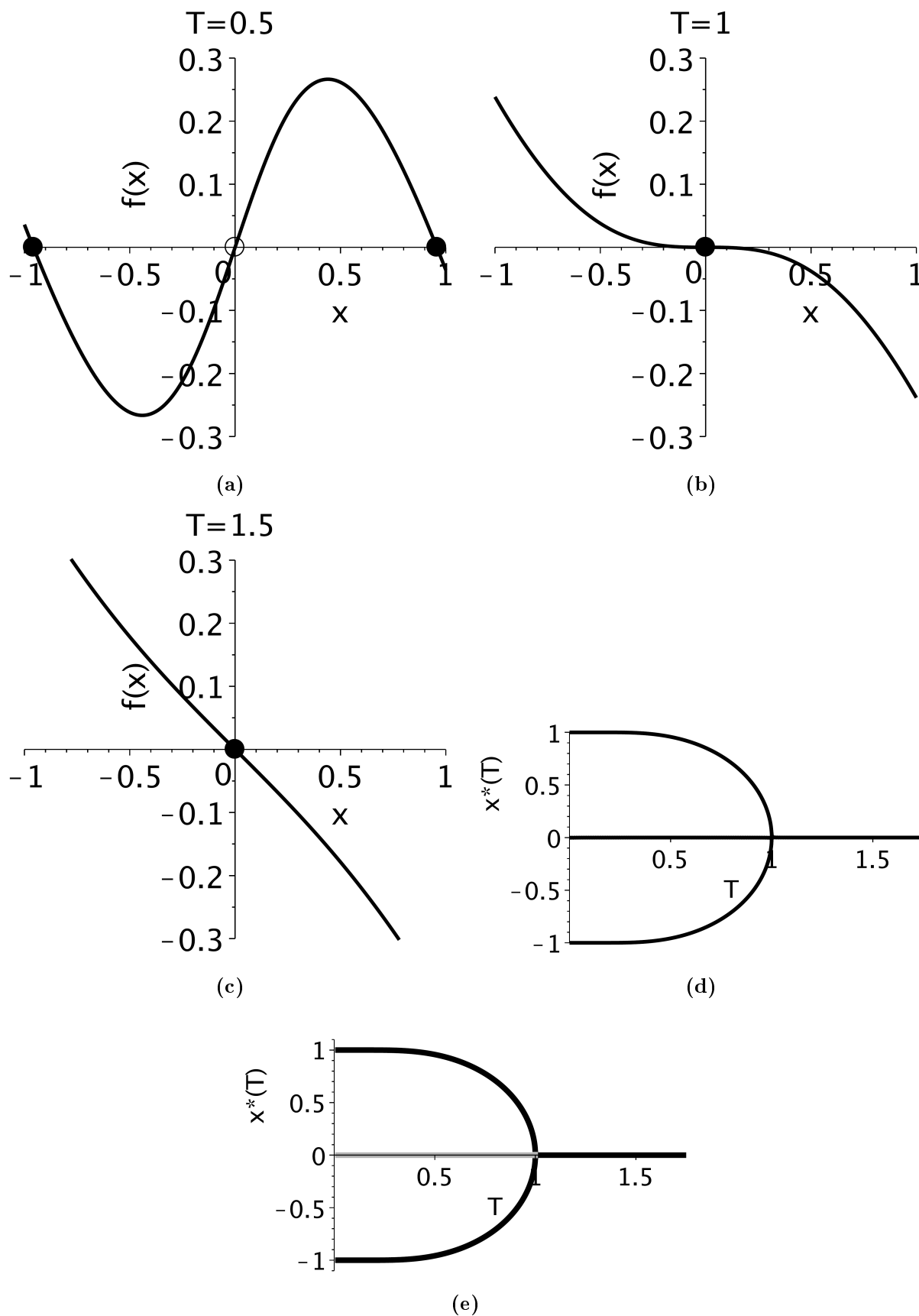


Figure 4.7: Pitchfork bifurcation in $\dot{x} = f(x, T) = -x + \tanh\left(\frac{x}{T}\right)$. (d) Bifurcation diagram obtained by straightforwardly using `implicitplot` (see text). (e) Bifurcation diagram with indication of the stability of the branches.

In Figure 4.7(a)-(c) we have plotted $f(x, T)$ for various values of T which reveal three fixed points at $0 < T < 1$ (two stable and one unstable), two of which are destroyed at $T = 1$ leaving one stable fixed points $x_*(T) = 0$ for $T \geq 1$. Such simultaneous creation/annihilation of two fixed points is characteristic of a pitchfork bifurcation. A convenient way to study the parameter dependence graphically is by using the plot command `animate`, see below. Figure 4.7 shows the bifurcation diagram. It is difficult to describe the branches by a simple analytical expression, since they are a solution to

$$x_* = \tanh\left(\frac{x_*}{T}\right)$$

so plotting a bifurcation diagram is not straightforward. But a fast way to do this is by making use of maple's `implicitplot`:



Maple

```
restart: with(plots):
f := x->-x + tanh(x/T); # define system
animate(plot, [f(x), x=-1..1], T=0.1..2.0, labels=["x", "f(x)"]);
implicitplot(f(x), T=0..2, x=-1..1, gridrefine=3, labels=["T", "x*(T)"]);
```

The `implicitplot` command plots the solution to $f(x, T) = 0$ within the supplied range of T and x , i.e. it shows the fixed points x_* as a function of T . Since there is no explicit expression, the method employs a grid to find a numerical solution. The default grid is rather coarse however, which is why we made use of the option `gridrefine=3`. This method using `implicitplot` is very powerful to get a quick glimpse of the fixed-point dependence on the external parameters (T in this case), but note that it does not provide information on the stability of the branches. Unfortunately, we have not been able to figure out a simple way in {maple} to draw the fixed point branches and color them by their stability characteristics. For lack of anything better, below we present a somewhat contrived workaround, but which seems to do the job nonetheless (see Figure 4.7)



Maple

```
restart: with(plots):
f := (x,T)->-x + tanh(x/T); # define system
df := D[1](f); # derivative to x
wu := proc(T,x) f(x,T)^(-sign(df(x,T))); end proc: # unstable branch
ws := proc(T,x) f(x,T)^( sign(df(x,T))); end proc: # stable branch
implicitplot([wu,ws], 0..1.75, -1..1, gridrefine=4, color=[green,red],
thickness=3, signchange=false);
```

In Figure 4.8 the numerical solution of $x(t)$ is plotted for a number of initial conditions, and two values of T . Such a figure can be made by using a loop, storing the plots and displaying them at the end:

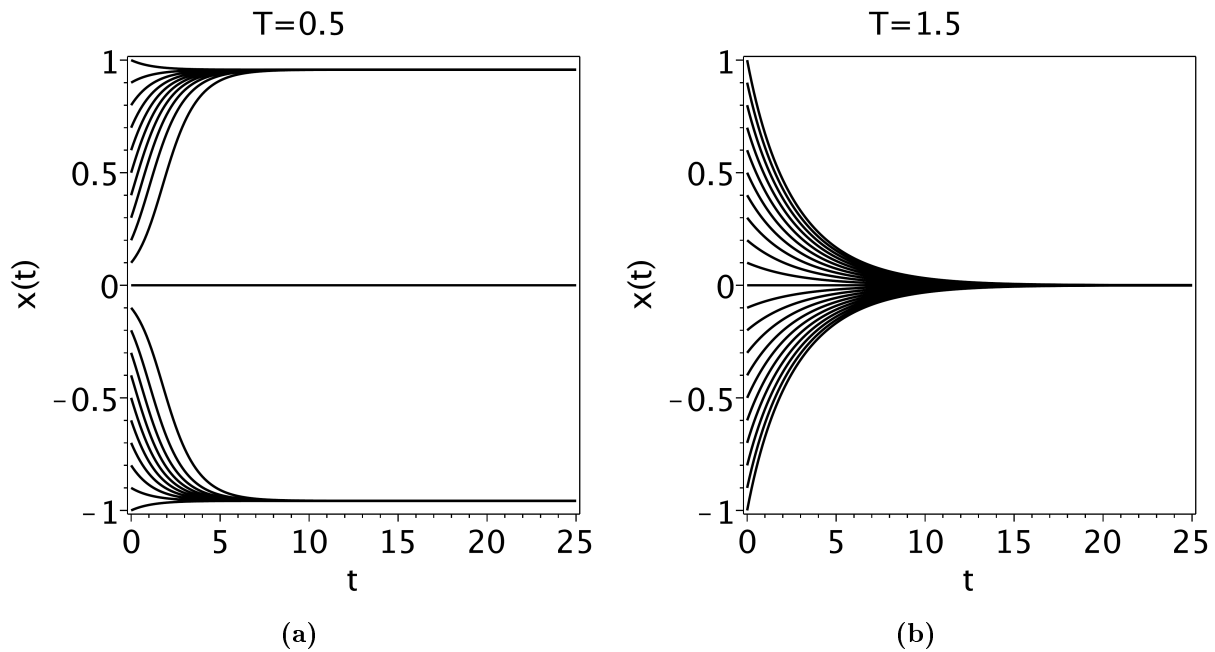


Figure 4.8: Evolution of the magnetization x for different values of the temperature T . In each panel the numerical solution is shown for 20 initial conditions.



Maple

```
restart; with(plots):
f := x->-x + tanh(x/T);           # define system
eq:=diff(x(t),t)=f(x(t));         # define differential eq
N := 20;                          # nr of initial cond.
T := 0.8;                          # set Temperature
for n from 0 to N do
  x0 := -1+2.0*n/N;               # set initial condition
  sol:=dsolve({eq,x(0)=x0},x(t),type=numeric); # calculate num. solution
  P[n] := odeplot(sol,[t,x(t)],0..30); # store the plot
end do;
display(seq(P[n],n=0..N),labels=["t","x(t)"]); # show all graphs
```

It is easy to verify that $x_* = 0$ is a fixed point for all values of T . The derivative of $f(x, T)$ is

$$f'(x) = -1 + [T \cosh^2(x/T)]^{-1}$$

so the stability of $x_* = 0$ is

$$\sigma = f'(0) = -1 + \frac{1}{T}$$

so $x_* = 0$ is stable if $T > 1$ and unstable if $T < 1$. The stability of the two other branches is less straightforward because we have no analytical expression for them. In the maple fragment below we calculate the numerical solution for $T = 0.5$ and evaluate the derivative to get the stability of the fixed point.

 Maple

```
restart; with(plots):
f := x->-x + tanh(x/T);
dfdx := D(f);                                # derivative of f
T := 0.5;                                    # set T
xfp := fsolve(f(x)=0,x=0..1,avoid={x=0});    # get the non-zero fixed point
sigma := dfdx(xfp);                          # evaluate deriv. at fxd point
```

This yields $x_* = 0.957\dots, \sigma = -0.833\dots$, showing that the fixed points is stable indeed at $T = 0.5$.

To see that the normal form of the bifurcation is $\dot{x} = rx - x^3$, we apply a Taylor approximation around the point of bifurcation, i.e. $T = 1, x = 0$. First we substitute $q = T - 1$ to translate the bifurcation to $q = 0$ which is more convenient. In **maple** it looks like

 Maple

```
restart; with(plots):
f := (x,q)->-x + tanh(x/(q+1));
f_taylor := mtaylor(f(x,q),[x=0,q=0],4);
collect(f_taylor,x);
```

which yields

$$\dot{x} = (-q + q^2)x - \frac{1}{3}x^3 + \dots$$

Upon substitution $z = x/\sqrt{3}$ and $r = q^2 - q$ one obtains $\dot{z} = rz - z^3$, which is the normal form of the pitchfork bifurcation, see [Figure 4.3](#).

4.6.1 Non-zero external magnetic field

The bifurcation characteristics change substantially when a non-zero external magnetic field is applied. Let us consider $m = 0.1$. Graphs of $f(x, T) = -x + \tanh\left(\frac{x+m}{T}\right)$ are shown in [Figure 4.9](#). The solutions to $f(x, T) = 0$, i.e. the fixed point branches $x_*(T)$, are shown in [Figure 4.9](#). It was again made with **implicitplot** together with the abovementioned trick to get information on the stability of the branches.

It is interesting to note that the fixed point branches in this case no longer display a pitchfork bifurcation as was the case for $m = 0$ (see [Fig Figure 4.7](#)). Rather, the upper branch, $x_*(T) > 0$, is no longer involved in any bifurcation. The lower two branches, on the other hand, now display a saddle node bifurcation, compare [Figure 4.4](#). Making a similar plot for $m = 0.005$ in [Figure 4.9](#) shows that as $m \rightarrow 0$ the saddle node bifurcation and the upper fixed point branch will 'melt' together, such that one will retrieve a pitchfork bifurcation at $m = 0$,

To locate the critical value of T_c at which the bifurcation takes place, we search the combination of x_c and T_c that satisfy

$$f(x_c, T_c) = 0 \quad \text{and} \quad f'(x_c, T_c) = 0$$

for which we use **fsolve**. See maple fragment below. For $m = 0.1$ this yields for example $T_c = 0.7287\dots, x_c = -0.520\dots$

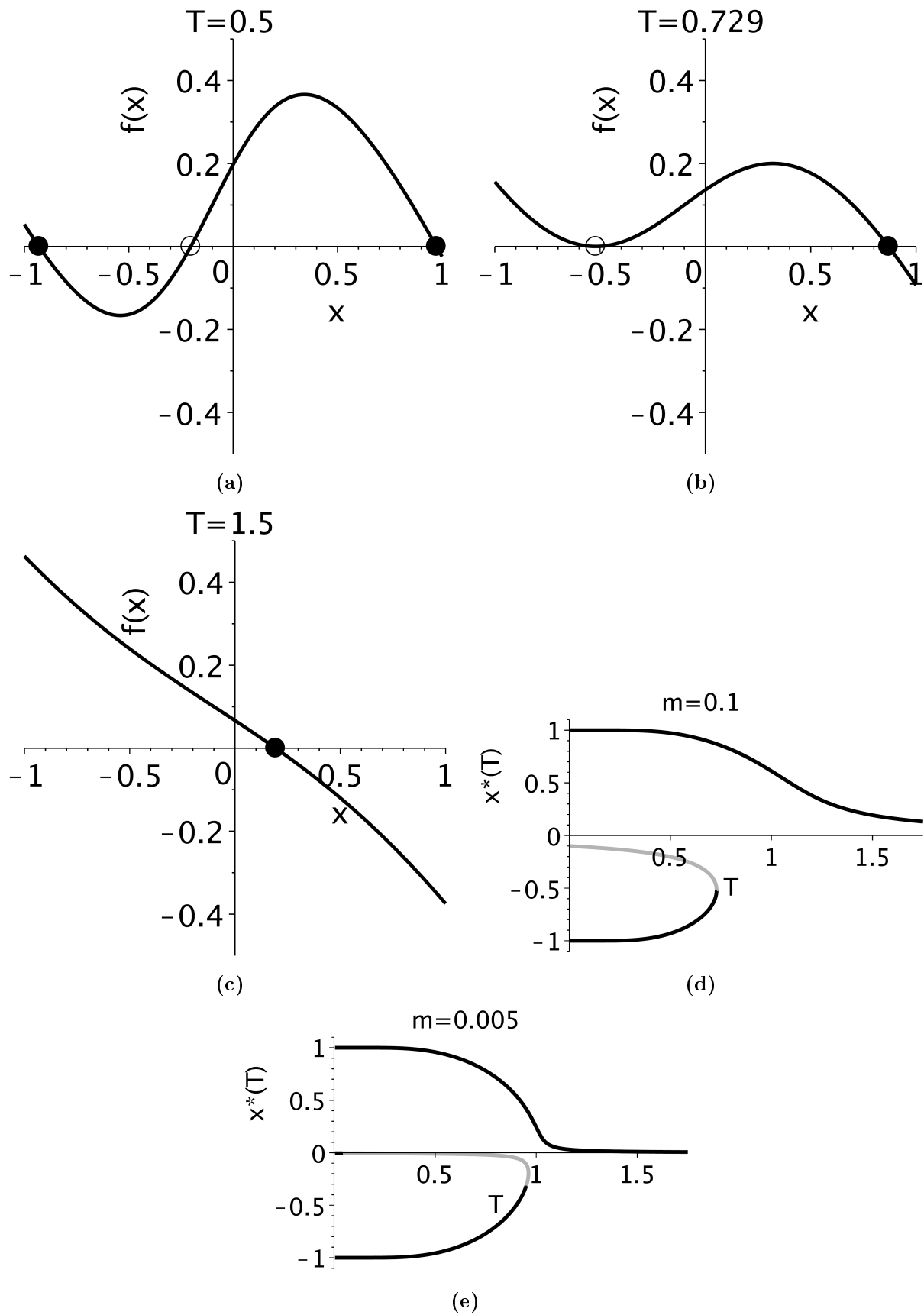


Figure 4.9: (a)-(c) Graphs of $f(x, T) = -x + \tanh\left(\frac{x+m}{T}\right)$ for an external magnetic field of $m = 0.1$. (d) Fixed point branches $x_*(T)$ for $m = 0.1$ and (e) for $m = 0.005$. Both cases display a saddle-node bifurcation, but as m becomes smaller, the diagram starts to more closely resemble that of a pitchfork bifurcation for $m = 0$.



Maple

```

restart: with(plots):
f := x->-x + tanh((x+m)/T);           # define system
df := D(f);                           # derivative
m := 0.1;                             # set external field
sol:=fsolve({f(x),df(x)},{x=-0.5,T=0.7}); # find (xc,Tc)

```

4.6.2 Example: Ferromagnetic system

As a model of a ferromagnet we conceive of a collection of N spins $s_i = -1, 1$ (i.e. down or up), $i = 1, \dots, N$. The spins influence each other's local magnetic field h

$$h_i = \frac{1}{N} \sum_{j=1}^N J_{ij} s_j + m \quad (4.12)$$

where the matrix coefficients J_{ij} express the connection strength between spin i and spin j ; m represents an externally applied magnetic field (if any). One could for example account for the distance between the spatial location $\mathbf{z}_i$ of the spins by taking $J_{ij} = f(|\mathbf{z}_i - \mathbf{z}_j|)$, but here we study the simplified situation where all connections are equal $J_{ij} = 1$.

The state of the system at time instance t is given by the vector $\mathbf{s}$. Given that each spins can attain one of two possible states, there are 2^N possibilities for $\mathbf{s}$. If the systems has temperature T , the probability of a particular state $\mathbf{s}$ is given by the Boltmann distribution

$$P(\mathbf{s}) \sim e^{-H(\mathbf{s})/T} \quad (4.13)$$

where $H(\mathbf{s})$ is the Hamiltonian $H(\mathbf{s}) = \frac{1}{2} \sum_{ij} s_i J_{ij} s_j$.

One can simulate the ferromagnet by 1) randomly picking a spin from the ensemble, say number i , 2) calculating the local field h_i by (4.12), and 3) flipping its state $s_i \rightarrow -s_i$ with probability $p(s_i \rightarrow -s_i) \sim \exp(-s_i h_i / T)$. One can prove that the distribution of states given by (4.13) will be satisfied if the transition probability is chosen as

$$p(s_i \rightarrow -s_i) = \frac{1}{2} \left[1 + \tanh \left(\frac{s_i h_i}{T} \right) \right] \quad (4.14)$$

In the limit $T \rightarrow 0$ this implies that if s_i is not aligned with its local field h_i , that is $h_i < 0$ while $s_i = +1$ or $h_i > 0$ while $s_i = -1$, then the probability of flipping s_i limits to 1, such that the spins will get aligned to the local field: $s_i \rightarrow \text{sign}(h_i)$. For $T > 0$, however, there is a non-zero probability that s_i and h_i remain disaligned.

The following maple program gives an implemtation that simulates a ferromagnetic system at predefined temperature T . In Figure 4.10 we give an example of the evolution of $N = 400$ spins from two different initial states for $T = 0.5$ and $T = 1.5$. The external magnetic field was set to $m = 0$. The initial state (see also the maple code) is created by setting all spins to $+1$ and flipping a preset fraction F of them.



Maple

```

restart; with(plots):
K := 20;
N := K*K;           # number of spins
show_state := proc(s) # show the current state
local i,j;
global K;
  listdensityplot([seq([seq(s[j+K*(i-1)],i=1..K)],j=1..K)],range=-1..1);
end proc;
choose_spin := 1 + rand(N): # choose spin at random
R := 10000;
random := evalf(rand(R)/R): # draw rnd nr from [0..1]

s := Array(1..N,1): # define spins and set to +1
F := 0.2;
for i from 1 to N do
  if( random() < F ) then # flip or not
    s[i] := -s[i];
  end if;
end do;
show_state(s);

m := 0; # set external field
T := 0.5; # set Temperature
iter := 10000; # set number of iterations

nrplots := 0;
for n from 0 to iter do # dynamics

  i := choose_spin();
  h_i := evalf(add(s[j],j=1..N)/N + m); # calc local field
  F := evalf((1 + tanh(s[i]*h_i/T))/2.0); # calc transition prob
  if( random() > F ) then s[i] := -s[i]; end if; # flip spin or not

  if( n mod 200 = 0 ) then
    nrplots := nrplots + 1;
    pl[nrplots] := show_state(s); # store some plots
  end if;

  x[n] := evalf(add(s[j],j=1..N)/N); # store mean state

end do;
# click in window to animate evolution of spins ...
display(seq(pl[k],k=1..nrplots),insequence=true);
# show mean state in time
plot([seq([n/N,x[n]],n=0..iter)],view=[0..iter/N,-1..1],labels=["t","x(t)"]);

```

To give a macroscopic description of the system, it is useful to calculate at each instance the mean magnetization of the spin system

$$x(t) = \frac{1}{N} \sum_{j=1}^N s_j(t)$$

which yields a value $\in [-1, 1]$. In the limit $N \rightarrow \infty$ one can derive that the dynamics of $x(t)$ is

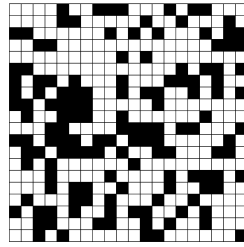


Figure 4.10: Evolution of the spin dynamics from two different initial states for two temperatures. The corresponding mean states are displayed in Figure 4.11.

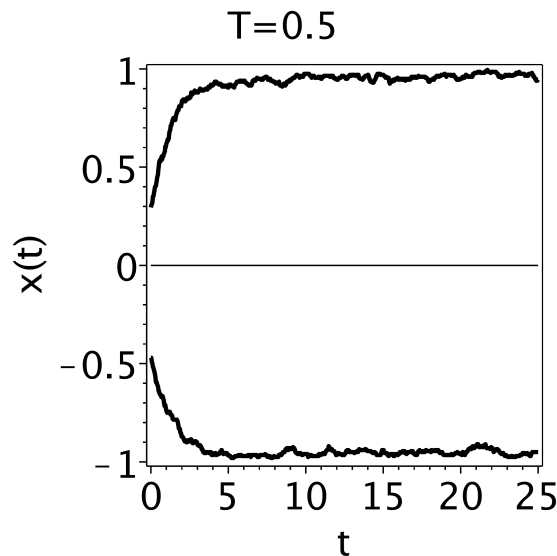


Figure 4.11: Evolution of the mean state x as a function of time $t = n/N$ for the cases shown in Figure 4.10. Compare also with Figure 4.8.

given by

$$\dot{x} = -x + \tanh\left(\frac{x + m}{T}\right) \quad (4.15)$$

In Figure 4.11 we display the evolution of x for the cases corresponding to Figure 4.10. Apart from so-called finite size fluctuations (i.e. N is not large enough), the evolution resembles that of Figure 4.8 reasonably well. An even better match will be obtained when the number of spins N is further increased.

4.7 Catastrophes, hysteresis, and phase transitions

A saddle node bifurcation is sometimes also called 'hard bifurcation', as opposed to transcritical and pitchfork bifurcations which are called 'soft bifurcation'. The reason for these terms can be understood by looking for instance at the saddle node bifurcation in figure Figure 4.9. If one starts at $T = 0$ at the lower branch ($x_* \simeq -1$) and one increases T in a slow manner such that the dynamical system has sufficient time to converge to the fixed point that belongs to the new temperature, one will move along the branch to the right. This can continue until $T_c = 0.7287\dots$ has been reached. Here the fixed point value is $x_* = x_c = -0.520\dots$. After a tiny increase of T beyond T_c , there is no longer a fixed point in the vicinity, so the system will rapidly move to

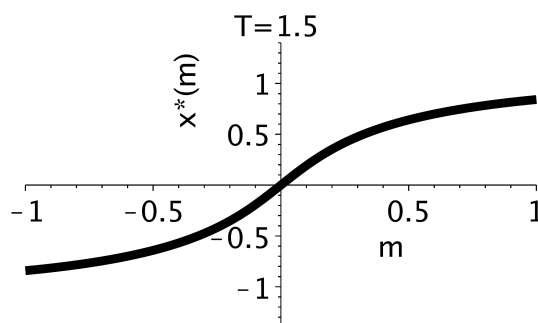


Figure 4.12: Left column: Fixed point branches as a function of the external magnetic field m for different values of the temperature T . The branches were determined using `implicitplot`. Unstable branches are colored gray. Right column: Fixed points determined by numerical integration of the governing equation $\dot{x} = -x + \tanh[(x + m)/T]$ each step using the previously computed fixed point as initial condition. In each of these panels first a forward sweep $m = -1..+1$ is conducted, followed by a backward sweep $m = +1..-1$. The plots illustrate the hysteresis effect when $T < 1$, together with the occurrence of 'hard' bifurcations as demonstrated by the sudden jumps to the other branch.

the upper fixed point branch. Usually this is a dramatic event for the system, sometimes referred to as 'catastrophe', which explains the name 'hard bifurcation'. In a pitchfork or transcritical bifurcation there is no need for a 'jump' to another branch because a smooth ('soft') transition to another branch is possible. Physicist tend to use the terms 'first order phase transition' and 'second order phase transition' for hard and soft bifurcations, respectively.

Let us reconsider the ferromagnetic system, but now use m as the bifurcation parameter instead of T . Using `implicitplot` we have plotted in the left column of Figure 4.12 the fixed-point branches as a function of m for various values of T . The branches for $T \geq 1$ appear not to be involved in any bifurcation. But the branches for $T < 1$ are much more interesting: as the branches seem to get 'folded', they exhibit two saddle node bifurcations. Apart from plotting these branches with `implicitplot`, it is illustrative to show the actual behaviour of the system by numerically integrating the system and computing its steady state point. These graphs are plotted in the right column of Figure 4.12. We have used the following `maple` code to obtain these plots.



Maple

```

restart; with(plots):
f := (x,m)->-x + tanh((x+m)/T);
eq:=diff(x(t),t)=f(x(t),m);
T := 0.5; mmin := -1.0; mmax := 1.0;
N := 200;
tinf := 100;                      # long time
ic := x(0) = -1;
for n from 0 to N do              # forward sweep from h = -1 to +1
    m := mmin + n*(mmax-mmin)/N:
    sol:=dsolve({eq,ic},x(t),type=numeric):
    X1[n] := subs(sol(tinf),x(t)); # large time solution
    M1[n] := m;
    ic := x(0) = X1[n];           # use previous value as initial cond.
od:
for n from 0 to N do              # backward sweep from h = +1 to -1
    m := mmax - n*(mmax-mmin)/N:
    sol:=dsolve({eq,ic},x(t),type=numeric):
    X2[n] := subs(sol(tinf),x(t)); # large time solution
    M2[n] := m;
    ic := x(0) = X2[n];           # use previous value as initial cond.
od:
p1 := plot([seq([M1[n],X1[n]],n=0..N)]): # plot forward sweep
p2 := plot([seq([M2[n],X2[n]],n=0..N)]): # plot backward sweep
display(p1,p2,view=[-1.1,-1.4..1.4],labels=["m","x*"]);

```

Note that we perform two sweeps. First we increase m from -1 to 1 in small steps and calculate at every step the 'large time' solution ($t = 100$ in this case). As initial condition we take the final value of x calculated in the previous step. In the second sweep the direction of m is reversed ($+1 \rightarrow -1$). The plots in the right column of Figure 4.12 nicely reveal two aspects of the system. Firstly, the sudden jumps, or 'catastrophes', that occur when $T < 1$, demonstrating the 'hard bifurcations' in the system. Secondly, one can notice *hysteresis*: the forward sweep is different from the backward sweep. By integrating the system one cannot show the location of the unstable branches, such as in the left column. On the other hand, it provides a good view of how the system really responds. Figure 4.13 combines the plots of the left column and right column of Figure 4.12 for the case of $T = 0.5$, illustrating the stable and unstable parts of the branch, the jumping from one branch to the other (hard bifurcations), and the hysteresis in the system. The right panel shows a three-dimensional view, illustrating the 'folding' of the fixed-point manifold that causes the saddle node bifurcations. The plot, including the shading, was made using one call to `implicitplot3d`:



Maple

```

restart; with(plots):
implicitplot3d(-x + tanh((x+m)/T),x=-1.1,T=0.5..1.5,m=-1.1..1.1,
    view=[-2.2,0.5..1.5,-1.4..1.4],
    grid=[50,50,50],style=surface,orientation=[-165,8,-122],
    style=surface,color=gray,light=[45,20,0.8,0.8,0.8]);

```

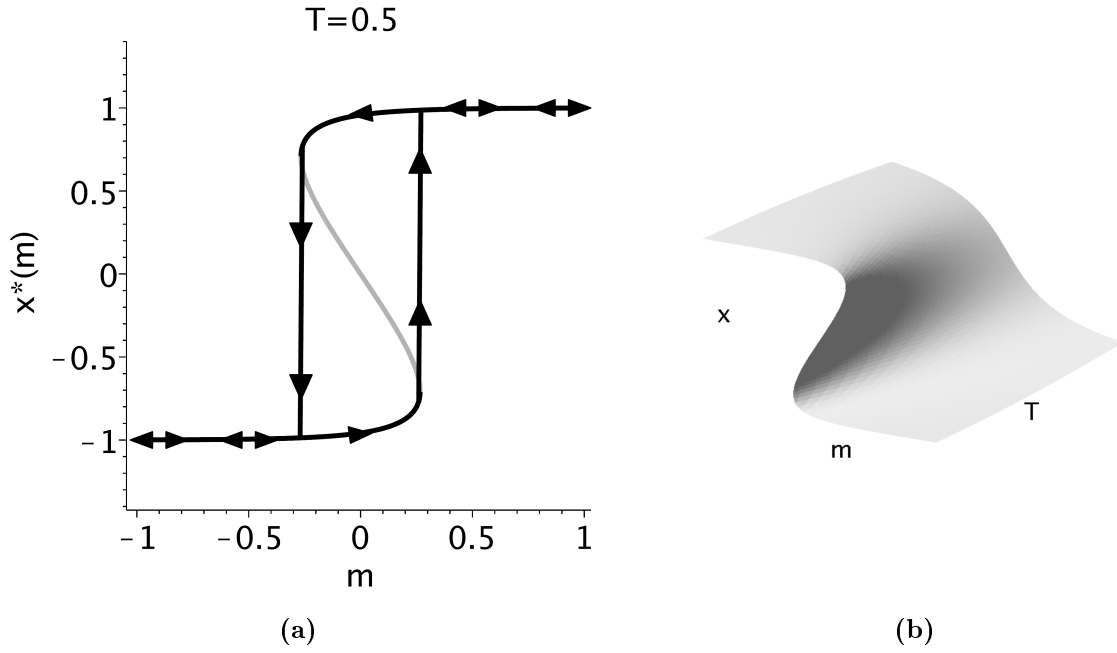


Figure 4.13: (a) Hysteresis and hard bifurcations in the ferromagnetic system with external field m . (b) 'Folding' of the fixed point manifold.

4.7.1 Example: multiple equilibria in the climate system

The earth's average temperature results from a balance between Q_{in} , the short-wave radiation received from the sun, and a loss Q_{out} due to long-wave radiation emitted by the earth. If R is the radius of the earth and πR^2 the area of the disk that is illuminated by the sun, the energy input to the earth per unit time is $Q_{\text{in}} = \pi R^2 S(1 - a)$, where $S = 1370 \text{ W/m}^2$ is the solar constant and a is the *albedo* of the earth, i.e. the reflection coefficient in the short-wave range. Due to the energy input the earth heats up and emits (long-wave) radiation along its entire surface: $Q_{\text{out}} = 4\pi R^2 \epsilon \sigma T^4$, where T is the temperature T of the earth in Kelvin, and $\sigma = 5.67 \cdot 10^{-8} \text{ kg s}^{-3} \text{ K}^{-4}$ denotes the constant of Stefan-Boltzmann. ϵ is the effective emissivity; for black body radiation $\epsilon = 1$, but here a 'gray body' approximation ($\epsilon < 1$) is more appropriate because of the presence of greenhouse gases in the atmosphere like CO_2 and water vapour, which absorb part of the long wave radiation and hence reduce the emission of radiation. The stationary mean temperature follows from the balance $Q_{\text{in}} = Q_{\text{out}}$.

If we account for the heat capacity of a surface layer with thickness h and specific heat c_v , we arrive at the following simple dynamical model

$$4\pi R^2 h \rho c_v \frac{dT}{dt} = [1 - a] \pi R^2 S - 4\pi R^2 \epsilon \sigma T^4$$

where ρ represents the density. Dividing by $4\pi R^2$ and defining $\beta = h \rho c_v$ gives

$$\beta \frac{dT}{dt} = [1 - a] \frac{S}{4} - \epsilon \sigma T^4 \quad (4.16)$$

Setting dT/dt to zero and solving for T one obtains the stationary temperature (fixed point)

$$T_* = \left[\frac{(1 - a)S}{4\epsilon\sigma} \right]^{1/4} \quad (4.17)$$

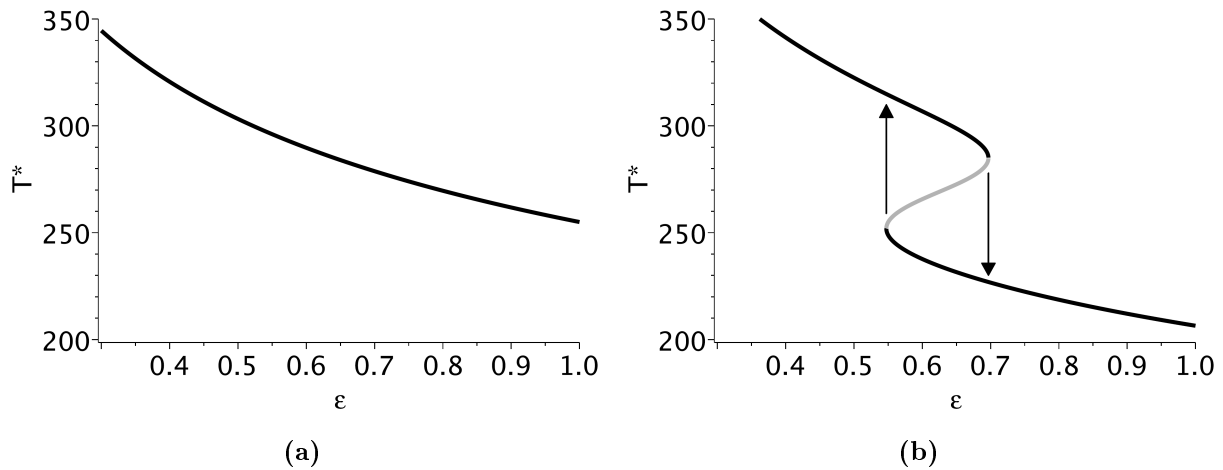


Figure 4.14: Average stationary temperature T_* of the earth as a function of the emissivity ϵ . Stable branches are black; unstable branches are gray. (a) Constant albedo $a = 0.3$. (b) Temperature dependent albedo.

With an average albedo of $a = 0.3$ and an emissivity of $\epsilon = 0.61$, one finds $T^* \simeq 289$ K, close to the current global average temperature. This simple model shows the effect of increased emissions of greenhouse gases as this lowers ϵ and hence increases the value of T^* ; see figure Figure 4.14. Note that the value of the parameters, i.e. a, ϵ and S uniquely determine the stationary state T^* (there is only one possible equilibrium). This is a stable equilibrium since the derivative in the fixed point T_* is negative

$$f'(T_*) = -4\epsilon\sigma T_*^3/\beta < 0$$

Below a maple implementation



Maple

```
restart: with(plots):
f := T -> ((1-a)*S/4-epsilon*sigma*T**4)/beta;
sigma := 5.67e-8; beta := 4e7; S := 1370;
a := 0.3; epsilon := 0.61;
# fixed point
Teq := fsolve(f(T)=0,T=0..350);
# stability, determine the derivative in the fixed point
df := D(f);
df(Teq);
# fixed point as a function of emissivity
epsilon := 'epsilon';
implicitplot(f(T),epsilon=0.3..1,T=100..400,gridrefine=3,labels=["epsilon","T*"],
  style=solid,color=black,thickness=2);
```

Multiple equilibria can occur when the so-called *albedo feedback* is taken into account. This involves an interesting dependence of the albedo a on temperature, because ice tends to reflect the incoming sunlight much more than ocean water does. This yields a positive feedback: when it is globally cold, ice sheets will prevail and much of the solar energy will be reflected, thus sustaining the cold climate. However, in a warm climate ocean water will prevail and much of the

energy is absorbed, thus sustaining the warm climate. This is called the *albedo-feedback*, and as we will see below, it forms the reason for the existence of multiple equilibria.

To account for the temperature dependence of the albedo, we take the following simple model

$$a(T) = 0.4 - 0.3 \tanh\left(\frac{T - T_m}{\Delta T}\right) \quad (4.18)$$

where T_m is the melting point of ice. By virtue of the $\tanh$ function and the value of ΔT , the albedo is a smooth function of temperature encompassing both the ice ($T < T_m$) and the liquid water state ($T > T_m$). For $T \ll T_m : a = 0.7$, whereas for $T \gg T_m : a = 0.1$. Inserting this albedo function in (4.16) and solving (numerically) for the fixed point(s) T_* reveals that for a range of ϵ multiple equilibria occur, see Figure 4.14. One also notes the unstable branch (gray color) and the possibility for hysteresis: if one gradually increases ϵ beyond 0.697, one drops to the lower (cold) branch. If one subsequently reduces ϵ one will stay on the lower branch until the critical value $\epsilon = 0.547$ is reached, beyond which the system jumps back to the upper (hot) branch.

The following `maple` fragment shows how the plot in Figure 4.14 was made and how the critical values for ϵ were determined. For additional coloring of the branches by their stability, one can use the example given in section 4.6.



Maple

```
restart: with(plots):
f := T -> ((1-a(T))*S/4-epsilon*sigma*T^4)/beta;
a := T -> 0.4 - 0.3*tanh((T-Tm)/DT);
sigma := 5.67e-8; beta := 4e7; S := 1370;
Tm := 273; DT := 20;
implicitplot(f(T),epsilon=0.3..1,T=100..400,gridrefine=3,labels=['epsilon',"T*
↔"]);
# determine the critical values of epsilon
df := D(f):
fsolve({f(T)=0,df(T)=0},{T=280,epsilon=0.6});
fsolve({f(T)=0,df(T)=0},{T=250,epsilon=0.4});
```

Table 4.1: Parameter values in the simple climate model.

symbol	value	unit
S	1370	W/m^2
σ	$5.67 \cdot 10^{-8}$	$\text{kg s}^{-3}\text{K}^{-4}$
ϵ	0.61	-
c_v	4000	$\text{JK}^{-1}\text{kg}^{-1}$
ρ	1000	kg m^{-3}
h	10	m
$\beta = h\rho c_v$	$4 \cdot 10^7$	$\text{Jm}^{-2}\text{K}^{-1}$
T_m	273	K
ΔT	20	K

4.8 Exercises

4.8.1 Hysteresis

Downloads: `Hysteresis.mws`

Study the equation

$$\dot{x} = rx + x^3 - x^5$$

where r is a parameter of the system. The system undergoes a subcritical bifurcation.

Analytical approach:

- Find an expression for the fixed points x^* for arbitrary r .
- Plot $x^*(r)$ as a function of r and give each branch a different color.
- Determine the stability of the branches.

Numerical approach:

- Take $r = 0.3$ and $x(0) = 0.1$; determine the numerical solution and plot the evolution of the system. Repeat this for $r = -0.1$.
- Create a bifurcation diagram by plotting the numerically computed 'large time solutions' as a function of $r \in [-0.5, 0.5]$. Increase r in small steps, calculate the corresponding 'large time solutions', and take as the initial conditions the equilibrium solution of the previous(!) step. In this way you simulate a *slow* change of the parameter r : the system has time to equilibrate to the fixed points belonging to the new value of r .
- Repeat the procedure of (e) for r *decreasing* from 0.5 to -0.5.
- Plot the results of (e) and (f) in the same graph, and give an explanation. You may want to include the branches obtained in (b) in the same plot.

4.8.2 Stability and bifurcations

Downloads: `PitchforkBifurcation.mws`

Consider the system

$$\dot{x} = \frac{(\nu + \frac{1}{2})x + (\nu - \frac{1}{2})x^3}{1 + x^2}$$

with $\nu \leq 1/2$ a parameter.

- Find the fixed points.
- Classify the stability of each fixed points in terms of ν .
- Does the system exhibit a bifurcation? If so, indicate for which value of ν , classify the kind of bifurcation, and plot a bifurcation diagram.
- Set $\nu = 1/4$ and determine with Maple the numerical solution for the initial conditions $x(0) = 10^{-4}$ and $x(0) = 3$ and plot the time evolution.

4.8.3 Stability and bifurcations (2)

Downloads: `StabilityBifurcationsExp.mws`

Consider the system

$$\dot{x} = f_r(x) \quad f_r(x) = -2 - x + r \exp(-x^2)$$

for $r \in [-10, 10]$.

- Find (numerically) all fixed points for $r = 3$,
- and classify their stability.
- Specify the attraction domains of each these fixed points.
- Plot $f_r(x)$ for different values of r and determine from the plots how many bifurcations the system undergoes within the range $r \in [-10, 10]$? Give a qualitative characterisation the type of bifurcations (saddle-node, transcritical, etc).
- Prove analytically that the system undergoes a bifurcation when the stable fixed-points have the values $x_1^* = -1 - \sqrt{2}/2$, or $x_2^* = -1 + \sqrt{2}/2$. Calculate the critical values r_1 and r_2 corresponding to the bifurcations at, respectively, x_1^* and x_2^* .
- Can this system exhibit hysteresis? If so, give a detailed description of the hysteresis cycle, for example by drawing a sketch (pen and paper) or by a numerical method.

4.8.4 Earth's heat balance

Downloads: `EarthHeatBalance.mw`

The latitude dependent temperature of the Earth results from a balance between short-wave radiation received from the sun, and a loss due to long-wave radiation. A simplified equation describing this process is

$$\dot{T} = -\epsilon(T + 273)^4 + w(T) \cos(\theta) \quad (4.19)$$

where T represents the temperature in Celcius and θ the latitude $\in [0, 90^\circ]$. Time has been rescaled by a characteristic timescale involving the heat capacity of the earth. The parameter ϵ denotes the effective long-wave emissivity. The term $w(T)$ incorporates the net absorption of short-wave radiation and depends on the surface properties (ice absorbs sunlight much less than water) and therefore on the local temperature. We approximate

$$w(T) = a + b \operatorname{erf}(cT)$$

with $\operatorname{erf}()$ the standard error-function, $a = 4.4 \cdot 10^9$, $b = 1.3 \cdot 10^9$, $c = 0.2$. Furthermore, $\epsilon = 0.5$.

- Calculate all fixed points and their stability for $\theta = 0^\circ$, $\theta = 45^\circ$ and $\theta = 60^\circ$. If there is more than one stable fixed point, specify the domain of attraction.
- Give a (concise) physical interpretation of the results of (a).
- Determine and plot a potential function $V(T)$ for $\theta = 45^\circ$.
- Plot the time evolution of the temperature at $\theta = 45^\circ$ for two initial conditions: $T(0) = -100^\circ C$ and $T(0) = +100^\circ C$. Take $t_\infty = 10^{-6}$ for the maximum time (recall that time was rescaled).
- Find graphically the latitude(s) for which a bifurcation occurs (just to get an indication). Next, specify the exact criterion, and calculate (numerically) from this the critical value(s) for θ . What is the type of the bifurcation?

5 Two-dimensional continuous systems

5.1 Introduction

In this chapter, we study systems of two simultaneous ODEs given by

$$\begin{aligned}\dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}\tag{5.1}$$

Here $x(t)$ and $y(t)$ are the dependent variables and t is the independent variable. Only autonomous systems will be considered, which explains why the functions f and g do not have a dependence on t .

Second order ODEs can be converted to a system of two first order ODEs. Consider the general second order ODE

$$\ddot{x} = f(x, \dot{x})$$

By introducing a new variable for the first derivative $y = \dot{x}$, a system of two first order ODEs is obtained:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= f(x, y)\end{aligned}$$

In general, any N -th order ODE can be converted to N simultaneous first order ODEs by introducing $N - 1$ new variables for all but the highest derivative.

A classical example of an autonomous system of two ODEs are the *Lotka-Volterra* equations

$$\begin{aligned}\dot{N} &= r_1 N - \theta_1 N P \\ \dot{P} &= r_2 N P - \theta_2 P.\end{aligned}$$

These equations have found wide-spread application in various fields, ranging from autocatalytic chemical reactions, *predator-prey* systems in biology/ecology [Case, 2000] and economical models. An amusing version involving the attraction between lovers is presented in [Strogatz, 1994]. Here, we will focus on the biological interpretation of the model, i.e. the predator-prey model.

The prey and predator populations are N and P , respectively. In absence of predators, the prey population grows exponentially. The rate of predation is modeled based on the meeting probability and is therefore proportional to the product of P and N . The predator birth rate is dependent on the availability of food and therefore also takes the form PN but with a different prefactor. Finally, the death rate of predators is proportional to P , as caused by either natural death or emigration.

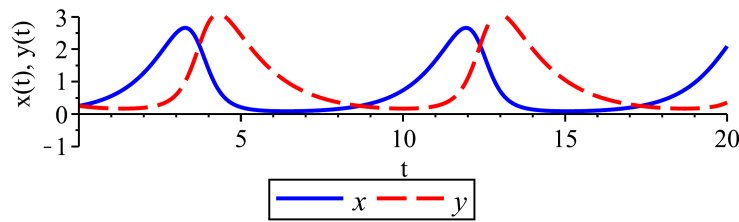


Figure 5.1: Timeseries of Lotka-Volterra system for $\alpha = 3/4$ and $x(0) = 0.25$, $y(0) = 0.25$.

There are four parameters in the problem above, which can be reduced to one when the equations are made dimensionless. Indeed, with the change of variables

$$x = \frac{r_2}{r_1}N, \quad y = \frac{\theta_1}{r_1}, \quad t' = r_1 t, \quad \alpha = \frac{\theta_1}{r_1}$$

the system simplifies to

$$\begin{aligned} \dot{x} &= x - xy \\ \dot{y} &= xy - \alpha y \end{aligned} \tag{5.2}$$

To get a feeling for the behaviour of this nonlinear system we perform numerical simulation with `maple`. The key commands are `dsolve` and `odeplot`, and the solution method is exactly identical to that of a 1-D system (see section XXX).



Maple

```
restart; with(DEtools); with(plots);
f := (x, y) -> x*(1-y);
g := (x, y) -> y*(x-alpha);
DE1 := diff(x(t), t) = f(x(t), y(t));
DE2 := diff(y(t), t) = g(x(t), y(t));
DE := DE1, DE2;
alpha := 3/4;
ic1 := x(0) = .25, y(0) = .25;
sol1 := dsolve({DE, ic1}, [x(t), y(t)], type = numeric, maxfun = -1);
tstart := 0; tend := 20;
odeplot(sol1, [ [t, x(t)], [t, y(t)] ], tstart .. tend,
        labels = ["t", "x(t), y(t)"], numpoints = 1000,
        legend = ['x', 'y']);
```

The result of the `maple` fragment, which uses $\alpha = 3/4$ and initial condition $x(0) = 0.25$, $y(0) = 0.25$, is shown in Figure 5.1. As can be seen, the predator- and prey-populations do not reach an equilibrium but instead vary periodically. This behaviour will be analysed in detail in subsequent sections.

5.2 The phase-space

The general 2-D system ((5.1)) can be written succinctly as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{5.3}$$

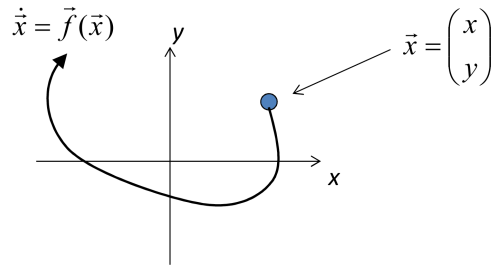


Figure 5.2: The system (5.3) causes the state vector $\mathbf{x}$ to change at a rate $\mathbf{f}$, thereby moving it through the phase space and tracing out a unique trajectory.

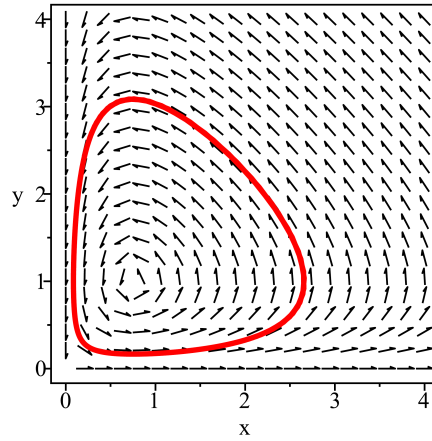


Figure 5.3: Phase portrait of a periodic solution of the Lotka-Volterra system.

where

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

The vector $\mathbf{x}$ is called the *state vector* of the system. The space spanned by the dependent variables x and y is called the *phase space*. Each point $\mathbf{x}$ in this space represents a unique state of the system. The function $\mathbf{f}$ will change the state of the system as time progress, thereby tracing out a trajectory $\mathbf{x}(t)$ in the phase space. This is shown schematically in Figure 5.2.

The phase space is an important concept in the study of dynamical systems, because it allows for a geometrical perspective on the system. Indeed, by comparing equation (5.3) to that describing a weightless particle (tracer) in a flow field $\mathbf{u}(x, y)$

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$$

it is immediately clear that $\mathbf{f} = \mathbf{u}$ and therefore we can interpret the ODE system as a point particle at location $\mathbf{x}$ which moves through the phase space with a velocity $\mathbf{f}(\mathbf{x})$.

When the system has a fixed point and the initial conditions are within the domain of attraction of that fixed point, the state vector will trace out a unique trajectory through the phase space which starts at the initial conditions and finds its way to the fixed point never to leave it again. Periodic solutions will trace out closed orbits in phase space.

The phase space representation of the timeseries shown in Figure 5.1 is a closed orbit (Figure 5.3). Hence, this is strong evidence that the solution is periodic. The arrows displayed in the plot are tangent vectors of $\mathbf{f}$ which indicate the direction of the trajectories (flow) in phase space.

This figure was created using the following `maple` fragment:



Maple

```
restart; with(DEtools); with(plots);
f := (x, y) -> x*(1-y);
g := (x, y) -> y*(x-alpha);
DE1 := diff(x(t), t) = f(x(t), y(t));
DE2 := diff(y(t), t) = g(x(t), y(t));
DE := DE1, DE2;
alpha := 3/4;
phaseportrait([DE], [x(t), y(t)], t = 0 .. 10, [[x(0) = .5, y(0) = .5]],
             stepsize = .1, x = 0 .. 4, y = 0 .. 4);
```

The workhorse in this fragment is the function `phaseportrait` from the `DEtools` toolbox. This command plots a grid of tangent vectors (flow direction), as well as one or more trajectories with differing initial conditions.

5.3 A brief review of linear 2-D systems

Before continuing the discussion on non-linear systems, a brief revision is required for linear systems of the form

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

where the coefficients a , b , c and d are assumed to be real-valued. This system can be written in a matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x} \quad \text{where } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A closed-form solution to this system can be found by diagonalization of A , which comprises the identification of an invertible coordinate transformation

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

for which the matrix A diagonalises:

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = S^{-1}AS \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

If this is the case, then $\tilde{x}$ and $\tilde{y}$ represent the coordinates in the eigenframe of this system. As before, we disregard the situation in which the matrix A is not diagonalizable, which occurs when $\sigma_1 = \sigma_2$. Note that in its diagonal form, the system is given by

$$\begin{aligned}\dot{\tilde{x}} &= \sigma_1 \tilde{x} \\ \dot{\tilde{y}} &= \sigma_2 \tilde{y}\end{aligned}$$

with solution

$$\tilde{x}(t) = \tilde{x}(0)e^{\sigma_1 t}, \quad \tilde{y}(t) = \tilde{y}(0)e^{\sigma_2 t} \quad (5.4)$$

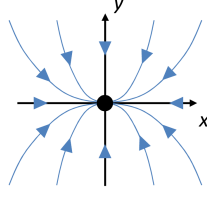


Figure 5.4: The behaviour of a dynamic system in the neighbourhood of a fixed point.

The eigenvalues σ_1 and σ_2 determine the character of the system. The characteristic polynomial of the eigenvalue problem $A\mathbf{x} = \sigma\mathbf{x}$ is

$$\sigma^2 - T\sigma + D = 0$$

Where $T = a + c$ and $D = ac - bd$ are the trace and determinant of A , respectively. The roots of this second order polynomial are given by

$$\sigma_{1,2} = \frac{1}{2} \left(T \pm \sqrt{T^2 - 4D} \right)$$

When the eigenvalues are real ($T^2 - 4D > 0$), three cases can be distinguished:

- $\sigma_1 < 0$ and $\sigma_2 < 0$. The fixed point is a *stable node* which all trajectories converge to. If $\sigma_1 = \sigma_2$, the system will move to the fixed point in straight lines. If $\sigma_1 > \sigma_2$, the trajectories will first move in the y_1 direction until y_1 is small enough;
- $\sigma_1 > 0$ and $\sigma_2 < 0$. This is a *saddle*. The fixed point attracts in the y_2 direction, but repels in the y_1 direction.
- $\sigma_1 > 0$ and $\sigma_2 > 0$. The fixed point is an *unstable node*.

When $T^2 - 4D < 0$, the eigenvalues will be imaginary, in which case oscillatory motion is to be expected. As the coefficients a , b , c and d are real, the eigenvalues will be a conjugate pair: if $\sigma_1 = \alpha + \beta i$ then $\sigma_2 = \alpha - \beta i$, or simply $\sigma_1^* = \sigma_2$. The character of the system is then determined by $\text{Re}(\sigma_1) = \text{Re}(\sigma_2) = \alpha$; the imaginary part determines the oscillation frequency only. Three cases can be distinguished:

- $\text{Re}(\sigma_1) = \text{Re}(\sigma_2) < 0$. The fixed point is an attractor, and trajectories describe a *stable spiral*;
- $\text{Re}(\sigma_1) = \text{Re}(\sigma_2) = 0$. The system has two identical eigenvalues and the system is therefore not diagonalizable. In this case, the fixed point is a *center*;
- $\text{Re}(\sigma_1) = \text{Re}(\sigma_2) > 0$. The system has an repellor, and trajectories describe a *unstable spiral*;

The eigenvalues of the 2-D matrix are related to T and D as:

$$\begin{aligned} T &= \sigma_1 + \sigma_2 \\ D &= \sigma_1 \sigma_2 \end{aligned}$$

This is useful for checking whether the eigenvalues are calculated properly. A systematic way to present the behaviour of 2-D systems is by plotting the behaviour as a function of T and D (Figure 5.5). Here we note that D is positive when both eigenvalues are positive or negative; hence the saddle will occur for $D < 0$. The distinction between a spiral and a node will depend on the value of the discriminant $T^2 - 4D$.

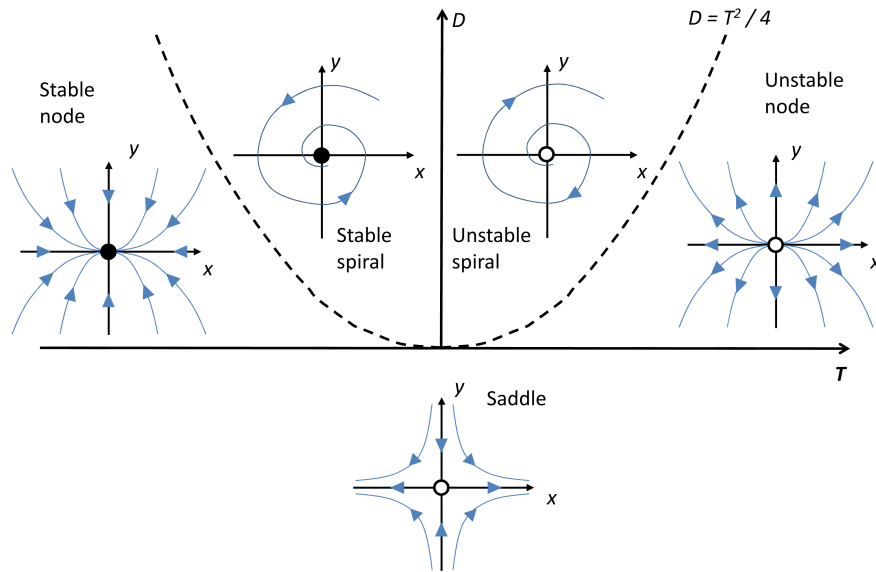


Figure 5.5: Classification of 2-D linear ODEs based on the trace T and determinant D of the matrix A .

5.4 Nonlinear systems

Linear systems are “uninteresting” in the sense that no matter where you start in the phase space, the system behavior is exactly identical and is governed by (5.4). For nonlinear systems, the dynamics are much richer. The behaviour of a nonlinear ODE in one region of the phase space can be entirely different from another region (Figure 5.6). Because the system of ODEs is nonlinear, the phase space may contain several fixed points each with their own dynamics; some may be attractors, some repellers, some may be saddles.

Fixed points (x^*, y^*) can be determined in the same way as for one-dimensional continuous systems, and need to satisfy $\dot{x} = 0$ and $\dot{y} = 0$, i.e.

$$\begin{aligned} f(x^*, y^*) &= 0 \\ g(x^*, y^*) &= 0 \end{aligned} \tag{5.5}$$

The lines for which $f(x, y) = 0$ and $g(x, y) = 0$ are called *nullclines* and fixed points exist on the intersection of both nullclines.

For the Lotka-Volterra equations (5.2), (5.2), the requirements of $\dot{x} = x - xy = 0$ and $\dot{y} = xy - \alpha y = 0$ lead to, respectively

$$\begin{aligned} \dot{x} = 0 &\rightarrow x = 0 \vee y = 1 \\ \dot{y} = 0 &\rightarrow x = \alpha \vee y = 0 \end{aligned}$$

and the system therefore has two fixed points $(x^*, y^*) = (0, 0)$ and $(x^*, y^*) = (\alpha, 1)$. The fragment below shows how to determine the fixed points in `maple`:

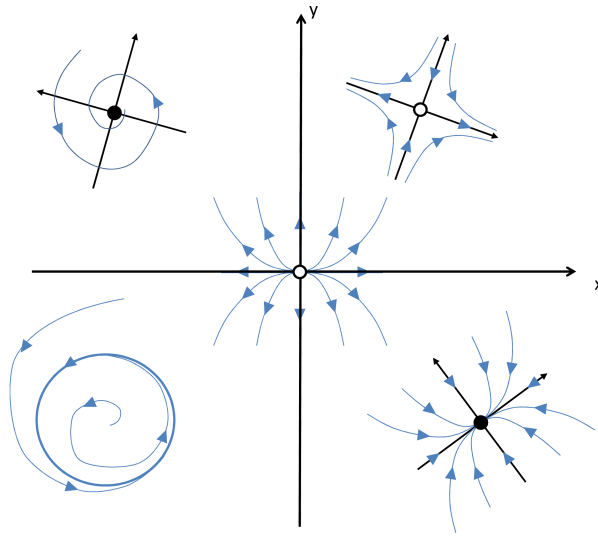


Figure 5.6: Example of the rich behavior of phase space for a nonlinear ODE.



Maple

```
restart; with(LinearAlgebra): with(plots):
f := (x,y) -> x*(1-y);
g := (x,y) -> y*(x-alpha);
fp := solve({f(x, y) = 0, g(x, y) = 0}, {x, y});
```

$$\mathbf{fp} := \{x = 0, y = 0\}, \{x = \alpha, y = 1\}$$

The stability of the fixed point (x^*, y^*) can be studied by perturbing it slightly and looking what happens: $x(t) = x^* + \delta(t)$, $y(t) = y^* + \epsilon(t)$

$$\begin{aligned}\dot{\delta} &= f(x^* + \delta, y^* + \epsilon) \\ &= f(x^*, y^*) + \frac{\partial f}{\partial x} \delta + \frac{\partial f}{\partial y} \epsilon + \mathcal{O}(\delta^2, \epsilon^2) \\ \dot{\epsilon} &= g(x^* + \delta, y^* + \epsilon) \\ &= g(x^*, y^*) + \frac{\partial g}{\partial x} \delta + \frac{\partial g}{\partial y} \epsilon + \mathcal{O}(\delta^2, \epsilon^2)\end{aligned}$$

Using ((5.5)) and neglecting terms of order δ^2, ϵ^2 one arrives at the equations governing the deviations from the fixed-point:

$$\begin{aligned}\dot{\delta} &= \frac{\partial f}{\partial x}(x^*, y^*) \delta + \frac{\partial f}{\partial y}(x^*, y^*) \epsilon \\ \dot{\epsilon} &= \frac{\partial g}{\partial x}(x^*, y^*) \delta + \frac{\partial g}{\partial y}(x^*, y^*) \epsilon\end{aligned}$$

We write this into a more formal form, by introducing the Jacobian $J(x, y)$ ¹

$$J(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

¹ Note that for a linear system, $J(x, y) = A = \text{constant}$.

which yields

$$\begin{pmatrix} \dot{\delta} \\ \dot{\epsilon} \end{pmatrix} = J(x^*, y^*) \begin{pmatrix} \delta \\ \epsilon \end{pmatrix}$$

Stability of the fixed point requires that deviations δ, ϵ vanish as t increases. As $J(x^*, y^*)$ is a constant matrix, it has a closed-form solution of the form (see section 5.3):

$$\tilde{\delta}(t) = \tilde{\delta}(0) e^{\sigma_1 t} \quad \tilde{\epsilon}(t) = \tilde{\epsilon}(0) e^{\sigma_2 t}$$

where $\tilde{\delta}$ and $\tilde{\epsilon}$ denote the coordinates in the eigendirections and $\sigma_{1,2}$ are the eigenvalues of $J(x^*, y^*)$. The different types of fixed points are exactly those presented in Figure 5.4, the only difference with linear systems being that the behaviour is limited to the local neighborhood of the fixed point.

The following `maple` fragment determines the stability of the two fixed points of the Lotka-Volterra equations (5.2),(5.2):



Maple

```
restart; with(LinearAlgebra): with(plots):
f := (x,y) -> x*(1-y);
g := (x,y) -> y*(x-alpha);
fp := solve({f(x, y) = 0, g(x, y) = 0}, {x, y});
jac := VectorCalculus[Jacobian]([f(x, y), g(x, y)], [x, y]);
ev1 := Eigenvalues(subs(fp[1], jac), output='list');
ev2 := Eigenvalues(subs(fp[2], jac), output='list');
```

$$\begin{aligned} \text{fp} &:= \{x = 0, y = 0\}, \{x = \alpha, y = 1\} \\ \text{ev1} &:= [1, -\alpha] \\ \text{ev2} &:= [-\sqrt{-\alpha}, \sqrt{-\alpha}] \end{aligned}$$

As $\alpha > 0$, the first fixed point is a *saddle* and the second a *center* (see Figure 5.4).

In non-linear systems, the stability of centers is fragile, as they are not attracting. Their behaviour can be easily destroyed by the nonlinearity. In the case of the Lotka-Volterra equations it is straightforward to show that the solutions are periodic because the system has a *constant of motion*

$$H = \frac{e^{x+y}}{x^\alpha y} \quad (5.6)$$

which remains constant under the action of the system for all time. This function has a minimum at the fixed point $(\alpha, 1)$, and increases monotonically away from the fixed points. The isolines of H are the trajectories of solutions to the ODE system. A systems that has a constant of motion are called a *conservative system* and will form the topic of the next section. Contours of H are plotted in Figure 5.7, which are also trajectory paths.

5.5 Conservative and gradient systems

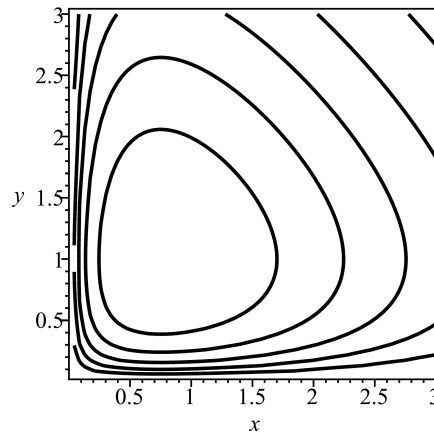


Figure 5.7: Isocontours of the constant of motion H for the Lotka-Volterra system.

5.5.1 Conservative systems

Conservative systems are systems which have at least one constant of motion or *conserved quantity*. Let us see what the requirements are for the ODE function $\mathbf{f}$ to have a constant of motion H . Because H needs to remain constant as the dynamical system traverses the phase space, we have $dH/dt = 0$ along trajectories and therefore

$$\dot{H} = \frac{d}{dt}H(x(t), y(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \mathbf{f} \cdot \nabla H = 0 \quad (5.7)$$

Using (5.7), it is straightforward to establish that (5.6) is indeed a constant of motion for the Lotka-Volterra system ((5.2)):

$$\mathbf{f} \cdot \nabla H = (x - xy) \frac{e^{x+y}}{x^\alpha y} \left(1 - \frac{\alpha}{x}\right) + (xy - \alpha y) \frac{e^{x+y}}{x^\alpha y} \left(1 - \frac{1}{y}\right) = 0$$

In `maple`, this can be checked as

 **Maple**

```
restart;
f := (x,y)-> x - x*y;
g := (x,y)-> x*y - alpha * y;
H := exp(x+y)/(x^alpha * y);
dHdt := diff(H, x)*f(x, y) + diff(H, y)*g(x, y);
simplify(dHdt);
```

$$dHdt := 0$$

The restriction $\mathbf{f} \cdot \nabla H = 0$ implies that $\mathbf{f}$ is everywhere perpendicular to ∇H (which in turn implies that $\mathbf{f}$ is everywhere tangential to isolines of H). Hence, if $f(x, y)$ and $g(x, y)$ take the form

$$\begin{aligned} f(x, y) &= \frac{1}{\rho(x, y)} \frac{\partial H}{\partial y} \\ g(x, y) &= -\frac{1}{\rho(x, y)} \frac{\partial H}{\partial x} \end{aligned} \quad (5.8)$$

where $\rho(x, y)$ is some weight function, then the system is conservative as can be demonstrated by substitution into (5.7). Note that ((5.8)) satisfy

$$\nabla \cdot (\rho \mathbf{f}) = 0 \quad (5.9)$$

Hence, if we can find a function $\rho(x, y)$ for which (5.9) holds, the system has a constant of motion².

Clearly, it is nontrivial to obtain $H(x, y)$ and/or $\rho(x, y)$ from the vector function $\mathbf{f}$, as (5.7) and (5.9) are partial differential equations. However, once either $H(x, y)$ or $\rho(x, y)$ is known, the other can be calculated using ((5.8)). For the Lotka-Volterra system, $\rho = -H/xy$.

The important class of *Hamiltonian systems* is obtained when the weight function $\rho(x, y) = 1$, for which ((5.8)) become

$$\begin{aligned} f(x, y) &= \frac{\partial H}{\partial y} \\ g(x, y) &= -\frac{\partial H}{\partial x} \end{aligned} \quad (5.10)$$

Hamiltonian systems have a central place in physics. The constant of motion H is called the *Hamiltonian* and represents the total energy of the system.

For Hamiltonian systems, (5.9) becomes

$$\nabla \cdot \mathbf{f} = 0 \quad (5.11)$$

which implies that *area* is conserved in the phase space. If we consider a small patch of the phase space and follow its evolution in time, it might deform into exotic shapes but its area always remains the same.

The fixed points of a Hamiltonian system are those points where

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0$$

i.e. where the energy surface H is entirely flat. As discussed, the trajectories of conservative systems are constrained to isolevels in H . If H has a local minimum (i.e. a fixed point), as plotted in Figure 5.8(a), the orbits will be closed and circle around the fixed point (Figure 5.8(a), black lines). When there is a local maximum in H , the contours of H will look identical to Figure 5.8(a), and we also expect closed periodic orbits. Note that the fixed point is non-attracting. When the fixed point is a saddle (Figure 5.8(b), black lines), the trajectories will escape the neighborhood of the fixed point along one of the diagonals.

² (5.9) has exactly the same form the equation for mass conservation in a compressible fluid, where ρ represents the fluid density of the fluid and $\mathbf{f}$ the velocity.

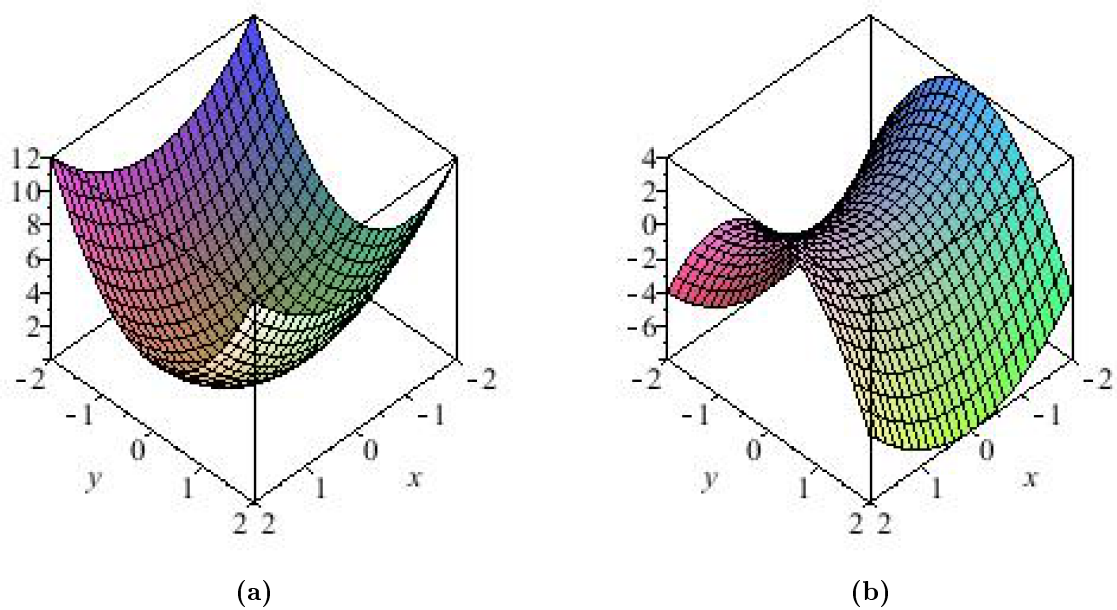


Figure 5.8: . Typical trajectories for conservative and gradient systems for a surface S (H , V for conservative and gradient systems, respectively).. Left figures: 3-D plot of S . Right figures: isolines of S (black lines; trajectories for conservative systems) and $-\nabla S$ (red vectors; direction field for gradient systems). (a) Conservative systems will circle around the fixed point, whereas gradient systems will find their way to the minimum. (b) Conservative systems escape to infinity across one of the diagonals, whereas gradient systems escape the neighborhood of the fixed point along one of the axes. (a) Local minimum. (b) Saddle.

**Example**

Consider the acceleration of a particle in a force field

$$m\ddot{x} = F(x)$$

where $F(x)$ is a *conservative force*, e.g. due to a magnetic field, a spring or a gravitational field. We will first demonstrate that this system is Hamiltonian and then calculate the H . Written as two first-order ODES, the system above is given by

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= F(x)/m\end{aligned}\tag{5.12}$$

and by using (5.11) we confirm that this is a Hamiltonian system. The Hamiltonian H can be found using ((5.10)):

$$\begin{aligned}H &= \int f(x, y)dy = \frac{1}{2}y^2 + c_1(x) \\ H &= - \int g(x, y)dx = V(x) + c_2(y)\end{aligned}$$

where we have introduced $F(x) = -m dV/dx$, $V(x)$ is a *potential*, and $c_1(x)$ and $c_2(y)$ are unknown functions. Combining the two equations, the Hamiltonian of this system is

$$H = V(x) + \frac{1}{2}y^2 + c$$

where c is a coefficient. Of course, the Hamiltonian is just the total energy per unit mass of the system: the first term is the potential energy per unit mass and the second term is the kinetic energy per unit mass.

**Example**

Consider the motion of a particle in a double-well potential given by $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ (and therefore $F(x) = -m(x^3 - x)$), for which the governing equations are (cf. (5.12), (5.12)):

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^3\end{aligned}\tag{5.13}$$

This system has three fixed points $(-1, 0)$, $(0, 0)$ and $(1, 0)$ which are denoted with open circles in Figure 5.9. A stability analysis reveals that these fixed points are a center, a saddle and a center, respectively. Also shown in Figure 5.9 are the contours of H , which are solutions to the ODE system. There are two minima in H at $(-1, 0)$ and $(1, 0)$, which are the energy wells. If the initial conditions are such that $H(t = 0) < 0$, the orbits are trapped in one of the wells. For initial conditions with $H(t = 0) > 0$, the system has too much energy to remain caught and orbits around both wells.

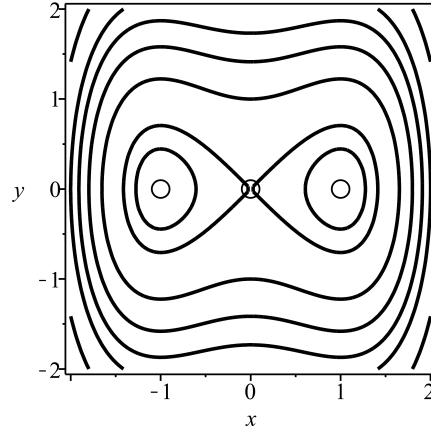


Figure 5.9: Contours of H for the double well system. As the system is conservative, these are also the trajectories in phase space. Open circles: fixed points of the system.

5.5.2 Gradient systems

As the name suggest, gradient systems are of the form

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x}) \quad (5.14)$$

where $V(x, y)$ is a *potential function*. In terms of our generic system ((5.1)), this means

$$\begin{aligned} f(x, y) &= -\frac{\partial V}{\partial x} \\ g(x, y) &= -\frac{\partial V}{\partial y} \end{aligned}$$

As for Hamiltonian systems, the fixed points are located where the potential V is entirely flat, i.e. where

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$$

By definition, the gradient of a function points in the direction of steepest ascent and is perpendicular to the isolines of V . Hence, if V has a local minimum, the system (5.14) will travel towards the fixed point following the steepest path down (Figure 5.8(a), red arrows). If the fixed point is a saddle, the system will escape the neighbourhood of the fixed point (Figure 5.8(b)).

Now, we ask ourselves the question: how does V change as a function of time for a specific trajectory $\mathbf{x}(t)$ governed by (5.14)?

$$\frac{d}{dt}V(\mathbf{x}(t)) = \nabla V \cdot \dot{\mathbf{x}} = -|\nabla V|^2$$

This is a powerful result: $|\nabla V|^2$ is negative by definition and therefore $\dot{V} \leq 0$ for *all* $\mathbf{x}$. Furthermore, $\dot{V} = 0$ implies that $|\nabla V| = 0$ and therefore indicates that $\dot{V} = 0$ occurs only at a fixed point $\mathbf{x} = \mathbf{x}^*$. If we further constrain ourselves to potential functions V that are single-valued, the only possible long-time behavior (assuming that the solution remains bounded) is that the solution is attracted to a fixed point³.

³ Assume that a closed orbit exists for which $\mathbf{x}(t+T) = \mathbf{x}(t)$ (periodic motion). If V is single-valued, this implies that $\int_0^T \frac{dV}{dt} dt = 0$. Substituting $ds = |\dot{\mathbf{x}}|dt$ and $\frac{dV}{dt} = \nabla V \cdot \dot{\mathbf{x}} = -\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$, we obtain that $\oint |\dot{\mathbf{x}}| ds = 0$. This can only occur when $|\dot{\mathbf{x}}| = 0$ for each point on the orbit. In this case, the entire “orbit” is a fixed “point” (e.g. $V(r) = 1 - r^2 + r^3$). We therefore conclude that closed orbits cannot exist when V is single-valued.

One can verify whether the functions $f(x, y)$ and $g(x, y)$ comprise a gradient system by checking if

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = -\frac{\partial}{\partial y} \frac{\partial V}{\partial x} + \frac{\partial}{\partial x} \frac{\partial V}{\partial y} = 0 \quad (5.15)$$

Note that all linear systems which have a symmetric coefficient matrix A , i.e. $f(x, y) = ax + by$, $g(x, y) = bx + cy$ can be written as a gradient system as is straightforward to verify with (5.15). For these systems, $V = -\frac{1}{2}ax^2 - \frac{1}{2}cy^2 - bxy$.



Example

Potential functions can be used to determine the stability of fixed points. As an example, consider a gradient system which has a potential function $V(x, y) = ax^4 + by^4$ where $a > 0$ and $b > 0$. In components, this system is given by

$$\begin{aligned}\dot{x} &= -4ax^3 \\ \dot{y} &= -4by^3\end{aligned}$$

The system has a single fixed point at the origin. The Jacobian is

$$J(x, y) = \begin{pmatrix} -12ax^2 & 0 \\ 0 & -12by^2 \end{pmatrix}$$

and the eigenvalues at the fixed point are $\sigma_1 = \sigma_2 = 0$. It is therefore not possible to decide the stability of the centers using the linearization technique. However, because the fixed point comprises a local minimum in the potential function V because $a > 0$ and $b > 0$, we conclude that the fixed point is stable.

**Example**

Situations in which V is multi-valued are not as exotic as it may seem at first sight. In fact one can find a classical example in the velocity field induced by a point vortex in an incompressible and irrotational fluid (i.e. a *potential flow*), in which case V is given by [Batchelor, 2000, White, 1999]

$$V = \tan^{-1} \left(\frac{y}{x} \right)$$

This is a fundamental building solution which is used e.g. to determine the lift on an airfoil [White, 1999]. Assume that a weightless particle which is advected by the point vortex, in which case (5.14) is given by

$$\begin{aligned} \dot{x} &= -\frac{y}{\sqrt{x^2 + y^2}} \\ \dot{y} &= \frac{x}{\sqrt{x^2 + y^2}} \end{aligned}$$

If we were to simulate the system above using `maple`, we would see that the solutions describe closed orbits in phase space. This is a direct consequence of the multi-valued nature of V . Indeed note that one of the transformation rules for the polar coordinates is $\tan \theta = y/x$ and therefore $V = \theta$. This function is therefore similar to a spiral staircase, and the system will go round and round indefinitely. This can also be seen when expressing the equations of motion in polar coordinates which results in $\dot{r} = 0$ and $\dot{\theta} = r^{-1}$. As the angular velocity $u_\theta = r\dot{\theta} = 1$ is constant, we conclude that the vortex causes all the particles to move at the same velocity, or in our analogy all particles travel down the spiral staircase at the same velocity.

5.6 Limit cycles

A new animal in the “zoo” of possibilities in nonlinear ODES is the *limit cycle*. Limit cycles occur in a wide variety of phenomena. Physically they represent self-sustained oscillations as occurring, for example, in meteorology, electronics, and in biological systems (heart-rhythms, body temperature).

A limit cycle has two defining properties:

- it is a closed orbit in the phase space;
- it is an attractor (or both a repeller and an attractor in exceptional cases).

Clearly, the closed orbits which occur for conservative systems such as the Lotka-Volterra equation are *not* limit cycles because they violate property 2.

An example of a limit cycle is shown in Figure 5.10. Two timeseries $x(t)$ starting with different initial conditions are plotted which were generated by numerically integrating the system

$$\begin{aligned} \dot{x} &= -y + \frac{x}{4\sqrt{x^2 + y^2}}(1 - x^2 - y^2) \\ \dot{y} &= x + \frac{y}{4\sqrt{x^2 + y^2}}(1 - x^2 - y^2) \end{aligned} \tag{5.16}$$

The `maple` fragment used to create Figure 5.10 is presented below.

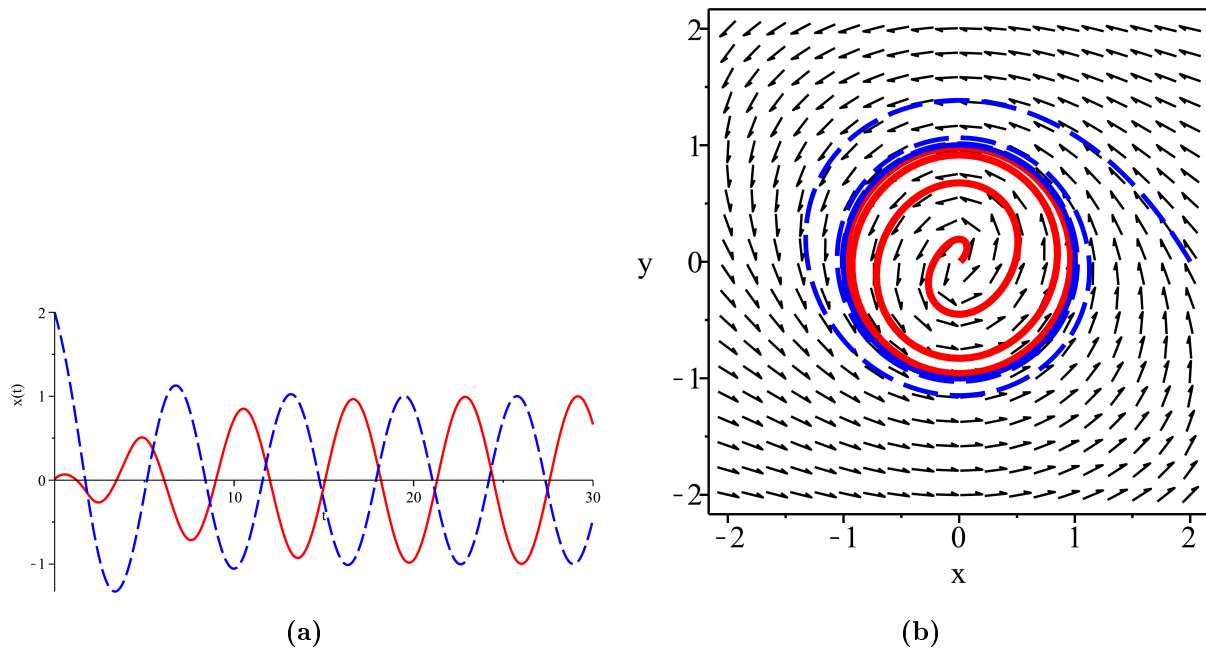


Figure 5.10: Limit cycle



Maple

```

restart; with(LinearAlgebra): with(DEtools):
f := (x,y) -> -y + x/(4*sqrt(x^2+y^2)) * (1-x^2-y^2);
g := (x,y) -> x + y/(4*sqrt(x^2+y^2)) * (1-x^2-y^2);
DE1 := diff(x(t),t) = f(x(t), y(t));
DE2 := diff(y(t),t) = g(x(t), y(t));
DE := DE1, DE2;
ic1 := x(0) = .01, y(0) = 0;
ic2 := x(0) = 2, y(0) = 0;
tstart:=0; tend:= 30;
phaseportrait([DE], [x(t), y(t)], t = tstart .. tend, [[ic1],[ic2]],
    stepsize = .1, x = -2 .. 2, y = -2 .. 2,
    color = black, linecolor = [red, green]);

```

After a short transient, the two time-series can be seen to display the same periodic motion (Figure 5.10(a)). Hence, the system forgets about its initial conditions, which is typical for all systems which have an attractor, and then remains locked in a periodic cycle. The limit cycle is perhaps even better discernable in the phase space (Figure 5.10(b)). The trajectory starting close to the origin (red solid line) can be seen to spiral outward initially as it is attracted to the limit cycle. The trajectory starting far from the the origin (blue dashed line) spirals inward and is rapidly attracted to the limit cycle.

In this particular case, the system ((5.16)) becomes much simpler when changing to polar coordinates (see Appendix C):

$$\begin{aligned}\dot{r} &= \frac{1-r^2}{4} \\ \dot{\theta} &= 1\end{aligned}$$

The point $r^* = 1$ is a stable fixed point of this system. Hence, the system will spiral outward

when $r(t = 0) < 1$ and inward for $r(t = 0) > 1$. As the phase angle θ increases continuously, we conclude we have identified a limit cycle.

**Example**

The *van der Pol* equation is a prototype system for self-excited oscillations given by

$$\ddot{x} + x = \mu(1 - x^2)\dot{x} \quad (5.17)$$

where the parameter $\mu > 0$. This equation was developed by Balthasar van der Pol in 1920 and describes the behavior of a triode circuit. The equations are those of an oscillator $\ddot{x} + k\dot{x} + x = 0$ with damping coefficient $k = -\mu(1 - x^2)$. Hence, for $|x| < 1$ the damping will be *negative* and we expect the system to be repelled from the origin in an unstable spiral. For $|x| > 1$ the damping will be positive and we expect a damped oscillation.

By introducing $y = \dot{x}$, (5.17) can be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu(1 - x^2)y \end{aligned}$$

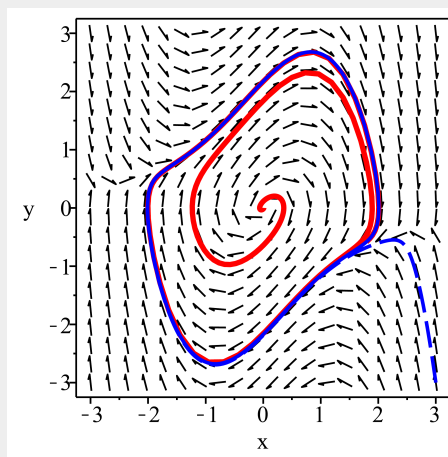


Figure 5.11: van der Pol equation, $\mu = 1$.

Figure 5.11 shows the phase portrait of the van der Pol equation for $\mu = 1$. One trajectory has initial conditions close to the origin and the other far away from it. Both trajectories are rapidly attracted to the limit cycle.

Note that it is impossible with our fixed point analysis to investigate the stability of the limit cycle because of its nonlocality. However, the fixed point at the origin does provide some information, albeit incomplete. The eigenvalues of the Jacobian at the $(0, 0)$ are

$$\sigma_{1,2} = \frac{1}{2}(\mu \pm \sqrt{\mu^2 - 4})$$

which indicates that the fixed point is an unstable spiral for $0 < \mu < 2$. Hence, the origin is a repeller which is an indication for the possible existence of a limit cycle (see also sections 5.7 and 5.7.1). For $\mu > 2$ the fixed point becomes an unstable saddle.

The van der Pol equation is a *Liénard system*, which is a class of dynamical systems which have a stable limit cycle surrounding the origin of the phase space. See [Strogatz1994, Verhulst1996] for more information.

5.7 Ruling out limit cycles - Lyapunov functions

Because limit cycles are not confined to a single point in the phase space, local techniques such as linear stability analysis cannot be used to prove their existence or determine their stability. Unfortunately, there are no general methods to prove the existence of limit cycles. However, there are methods that are able to confirm whether or not limit cycles exist for specific classes of systems.

Conservative systems cannot have limit cycles. All point within a limit cycle's basin of attraction will end up on the limit cycle eventually. For a conservative system this means that H is constant within the basin of attraction. However, because the motion is dependent on gradients in H , cf. ((5.8)) we conclude that the basin of attraction cannot occupy any area and needs to be constrained to a line or point⁴.

Gradient systems cannot have limit cycles because $dV/dt \leq 0$ for all initial conditions. Hence, after a sufficiently long time, the system is either in a fixed point (if V is single-valued) or may be on a non-attracting closed orbit (if V is multi-valued).

A *Lyapunov function* is similar to a potential functions of gradient systems, but are more widely applicable because the restrictions on V are much weaker. They are particularly useful to rule out the existence of limit cycles, but can also be used to demonstrate the stability of fixed points.

A Lyapunov function V for a dynamical system is each function that satisfies the properties

- $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for almost all other $\mathbf{x}$;
- $dV/dt < 0$ for almost all $\mathbf{x}$.

Gradient system satisfies these requirements (see section 5.5.2), as long as the potential function V is single-valued and $V(\mathbf{x}) > 0$ for all values of $\mathbf{x}$ except the fixed point (i.e. the trajectories cannot escape to infinity). However, Lyapunov functions are applicable to a much wider range of problems because it does not have the constraint that the dynamics are governed by V , unlike gradient systems,

There is no general way of obtaining Lyapunov functions, and consequently they are often obtained by trying informed guesses, such as

- $V(\mathbf{x}) = \|\mathbf{x}\|^2$
- $V(\mathbf{x}) = ax^2 + by^2$

When considering mechanical systems, a natural candidate for a Lyapunov function is the total energy in the system. For example, the governing equations for a spring-mass system including a quadratic damping term with friction coefficient μ are

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x - \frac{\mu}{m}|y|y\end{aligned}$$

where x is the displacement, y the velocity, k is the spring coefficient and m is the mass. The total energy per unit mass in this system is $\frac{1}{2}\frac{k}{m}x^2 + \frac{1}{2}y^2$. Let us consider whether $V = \frac{1}{2}\frac{k}{m}x^2 + \frac{1}{2}y^2$ is a suitable Lyapunov function. First, $V(\mathbf{x}^*) = 0$ which is easily shown since the only stable fixed point is $(0,0)$. Furthermore $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. Lastly,

$$\frac{dV}{dt} = \mathbf{f} \cdot \nabla V = y\frac{k}{m}x + \left(-\frac{k}{m}x - \frac{\mu}{m}|y|y\right)y = -\frac{\mu}{m}|y|y^2$$

⁴ A more rigorous argument can be made using the property $\nabla \cdot \rho \mathbf{f} = 0$.

which is ≤ 0 everywhere and therefore, V is a Lyapunov function.

Based on the fact that this system has a Lyapunov function, we conclude that no limit cycles exist. Furthermore, we conclude that $\mathbf{x}^* = 0$ is a stable node.

5.7.1 Is chaos possible in 2-D systems?

The short answer to this question is no. There are two types of long-term behaviour in 2-D systems:

- a fixed point;
- a closed orbit (either limit cycle or non-attracting).

The fact that no chaos can occur in 2-D systems follows from the *Poincaré-Bendixson* theorem, which can be used to prove the existence of limit cycles. The theorem considers a part of the phase space Ω which does not contain fixed points but does contain at least one trajectory which is confined within Ω . If this is the case, then this trajectory is a *closed* orbit, because it is impossible for trajectories to cross (this would indicate that the solutions were not unique, because one point in the phase space then has multiple values for $\mathbf{f}$).

The theorem in itself is not very practical because it is nontrivial to prove that trajectories remain within Ω . However, as the theorem proves that the only attractor which occupies a region in the phase space is a limit cycle, it automatically rules out chaos (more formally: no strange attractor can exist in 2-D, see section XXX).

We emphasize that this result is only valid for planar continuous 2-D systems. Indeed, chaos is possible in discrete systems with only one degree of freedom, and if the phase space is a torus, non-periodic motion is possible.

5.8 Bifurcations in 2-D systems

As discussed in the previous chapter, bifurcations in 2-D are changes in the behaviour of a dynamical system upon variation of the control parameter μ . More specifically, bifurcations occur when fixed points come into existence, when fixed points and/or limit cycles collide or when the stability of a fixed point changes.

The bifurcations identified in the previous chapter can all occur in 2-D as well. These bifurcations are discussed in section 5.8.1, and we will see that these are all associated with strictly real eigenvalues. A new bifurcation in 2-D is the Hopf-bifurcation (5.8.6), which is associated with imaginary eigenvalues. When limit cycles come into existence, or collide with fixed points, global bifurcations can occur (5.8.7).

5.8.1 Familiar bifurcations

The bifurcations which also exist in 1-D all share the property that the eigenvalues are *strictly real*.

5.8.2 Saddle-node bifurcation

The normal form of the saddle-node bifurcation is

$$\begin{aligned}\dot{x} &= \mu - x^2 \\ \dot{y} &= -y\end{aligned}$$

The characteristic feature of this bifurcation is that for $\mu < 0$ there are no fixed points, whilst two fixed points $x^* = -\sqrt{\mu}$ and $x^* = \sqrt{\mu}$ exist for $\mu > 0$. The Jacobian matrix is given by

$$J(x, y) = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

and since $J(x, y)$ is in diagonal form, the eigenvalues are $\sigma_1 = -2x$ and $\sigma_2 = -1$. The character of the two fixed points are summarised in the table below.

fixed point	σ_1	σ_2	$\mu < 0$	$\mu > 0$
$x^* = -\sqrt{\mu}, y^* = 0$	$-2\sqrt{\mu}$	-1	non-existent	stable node
$x^* = \sqrt{\mu}, y^* = 0$	$2\sqrt{\mu}$	-1	non-existent	saddle

5.8.3 Transcritical bifurcation

The normal form of the transcritical bifurcation is

$$\begin{aligned}\dot{x} &= \mu x - x^2 \\ \dot{y} &= -y\end{aligned}$$

The characteristic feature of this bifurcation is there are two fixed points which exchange their stability property at $\mu = 0$. The two fixed points are $x^* = 0$ and $x^* = \mu$ and the Jacobian matrix is given by

$$J(x, y) = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

and the eigenvalues are $\sigma_1 = \mu - 2x$ and $\sigma_2 = -1$. The character of the two fixed points are summarised in the table below.

fixed point	σ_1	σ_2	$\mu < 0$	$\mu > 0$
$x^* = 0, y^* = 0$	μ	-1	stable node	saddle
$x^* = \mu, y^* = 0$	$-\mu$	-1	saddle	stable node

5.8.4 Supercritical pitchfork bifurcation

The normal form of the supercritical pitchfork bifurcation is

$$\begin{aligned}\dot{x} &= \mu x - x^3 \\ \dot{y} &= -y\end{aligned}$$

The characteristic feature of this bifurcation is that there exists one stable fixed point for $\mu < 0$ which becomes unstable for $\mu > 0$, and two further stable fixed points which come into existence for $\mu > 0$. The three fixed points are $x^* = 0$, $x^* = \sqrt{\mu}$ and $x^* = -\sqrt{\mu}$ and the Jacobian matrix is given by

$$J(x, y) = \begin{pmatrix} \mu - 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

and the eigenvalues are $\sigma_1 = \mu - 3x^2$ and $\sigma_2 = -1$. The character of the three fixed points are summarised in the table below.

fixed point	σ_1	σ_2	$\mu < 0$	$\mu > 0$
$x^* = 0, y^* = 0$	μ	-1	stable node	saddle
$x^* = -\sqrt{\mu}, y^* = 0$	-2μ	-1	non-existent	stable node
$x^* = \sqrt{\mu}, y^* = 0$	-2μ	-1	non-existent	stable node

5.8.5 Subcritical pitchfork bifurcation

The normal form of the transcritical bifurcation is very similar to the supercritical version, except for the sign of the cubic:

$$\begin{aligned}\dot{x} &= \mu x + x^3 \\ \dot{y} &= -y\end{aligned}$$

The characteristic feature of this bifurcation is that there are two stable and one unstable fixed point for $\mu < 0$ and one unstable fixed point for $\mu > 0$. As for the supercritical pitchfork bifurcation, the three fixed points are $x^* = 0$, $x^* = i\sqrt{\mu}$ and $-i\sqrt{\mu}$ and the Jacobian is given by

$$J(x, y) = \begin{pmatrix} \mu + 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

and the eigenvalues are $\sigma_1 = \mu + 3x^2$ and $\sigma_2 = -1$. The character of the three fixed points are summarised in the table below.

fixed point	σ_1	σ_2	$\mu < 0$	$\mu > 0$
$x^* = 0, y^* = 0$	μ	-1	stable node	saddle
$x^* = -i\sqrt{\mu}, y^* = 0$	-2μ	-1	saddle	non-existent
$x^* = i\sqrt{\mu}, y^* = 0$	-2μ	-1	saddle	non-existent

The “proper” way of analysing bifurcations is by reduction to normal forms, as was done for 1-D systems in section XXX. However, the mathematics become quite complex and are beyond the scope of this book. Please see [Crawford1991, Verhulst1996] for more information.

We will avoid some of the mathematical complexity by making use of `maple`'s versatility. In particular, by plotting the behaviour of the fixed points and the eigenvalues as a function of the control parameter μ , it is possible to determine for which μ bifurcations occur and to identify their type.

**Example**

Consider the system

$$\begin{aligned}\dot{x} &= (\mu - 2)x + y + \sin(x) \\ \dot{y} &= x - y\end{aligned}\tag{5.18}$$

We are interested in the behaviour of the fixed points in this system as a function of the parameter μ . By implementing the system in `maple` and using `phaseportrait` to visualise the system behavior for various values of μ , it becomes clear that there is a bifurcation at $\mu = 0$ (Figure 5.12(a, b)).

To explore this in more detail, we would like to plot the fixed points (x^*, y^*) as a function of μ . Although it is in principle possible `implicitplot3d` to plot the roots of the above equation, note that the nullcline for $\dot{y} = 0$ is the diagonal $y = x$. Therefore, the governing equation for x^* is

$$(\mu - 1)x^* + \sin(x^*) = 0$$

This equation is not invertible, but can straightforwardly visualised using `implicitplot`. The fixed point $x^* = 0$ is a solution for all μ and the eigenvalues of $J(0, 0)$ are

$$\sigma_{1,2} := -1 + \frac{1}{2}\mu \pm \frac{1}{2}(4 + \mu^2)$$

The single fixed point $(0, 0)$ for $\mu < 0$ can be seen to bifurcate into three fixed points at $\mu = 0$ (Figure 5.12(c)). Judging from the eigenvalues of the Jacobian (Figure 5.12(d)), the fixed point changes from a stable node to a saddle node at $\mu = 0$. Hence, a *supercritical pitchfork bifurcation* occurs at $\mu = 0$. The `maple` code to generate these plots is given below.

```
restart; with(plots): with(LinearAlgebra):
f := (x,y) -> (mu-2) * x + y + sin(x);
g := (x,y) -> x - y;
implicitplot(subs(y=x, f(x,y)), mu=-1..2, x=-10..10, gridrefine=4,
             color=black, thickness=2,
             labeldirections=[horizontal, vertical]);
jac := VectorCalculus[Jacobian]([f(x,y), g(x,y)], [x, y]);
ev := Eigenvalues(subs(x=0, jac), output='list');
plot([Re(ev[1]), Re(ev[2])], mu = -1..2, thickness=2,
     labels=['mu', "Re( sigma )"], legend=['sigma[1]', 'sigma[2]'],
     color = black, linestyle=[solid, dash],
     labeldirections=[horizontal, vertical]);
```

5.8.6 Hopf bifurcations

A new bifurcation in 2-D is the *Hopf bifurcation*. This bifurcation involves the transition from a stable spiral ($\text{Re}(\sigma) < 0$) to an unstable spiral ($\text{Re}(\sigma) > 0$), and can be linked to the "birth of a limit cycle".

Consider the system:

$$\begin{aligned}\dot{x} &= -y - x(-\mu + x^2 + y^2) \\ \dot{y} &= x - y(-\mu + x^2 + y^2)\end{aligned}\tag{5.19}$$

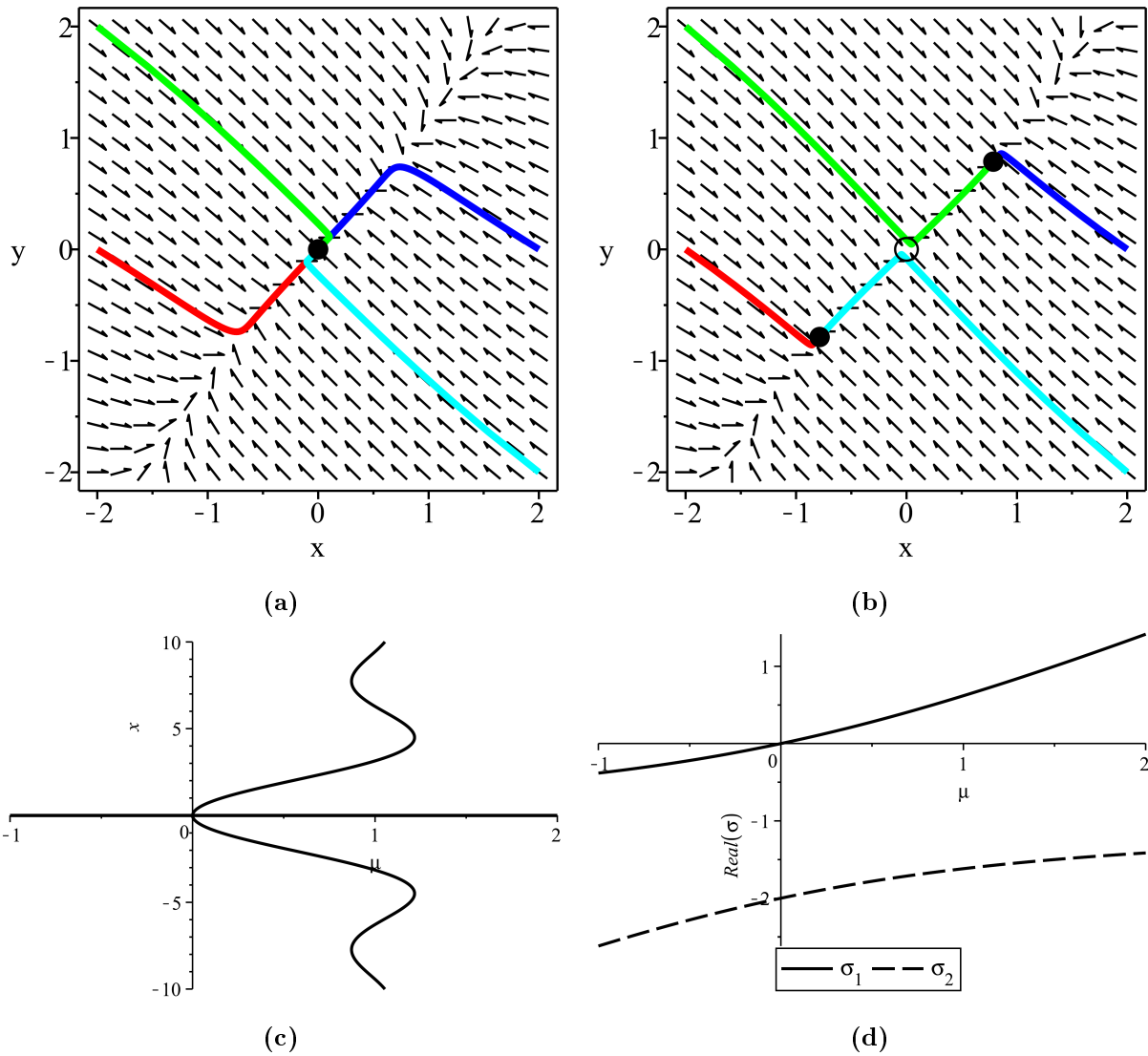


Figure 5.12: Supercritical pitchfork bifurcation for the system ((5.18)). (a, b) phase portraits for $\mu = -0.1$ and $\mu = 0.1$; c) x^* as a function of μ ; d) the eigenvalues (which are purely real in this case) for the fixed point $(0, 0)$. (a) $\mu = -0.1$. (b) $\mu = 0.1$.

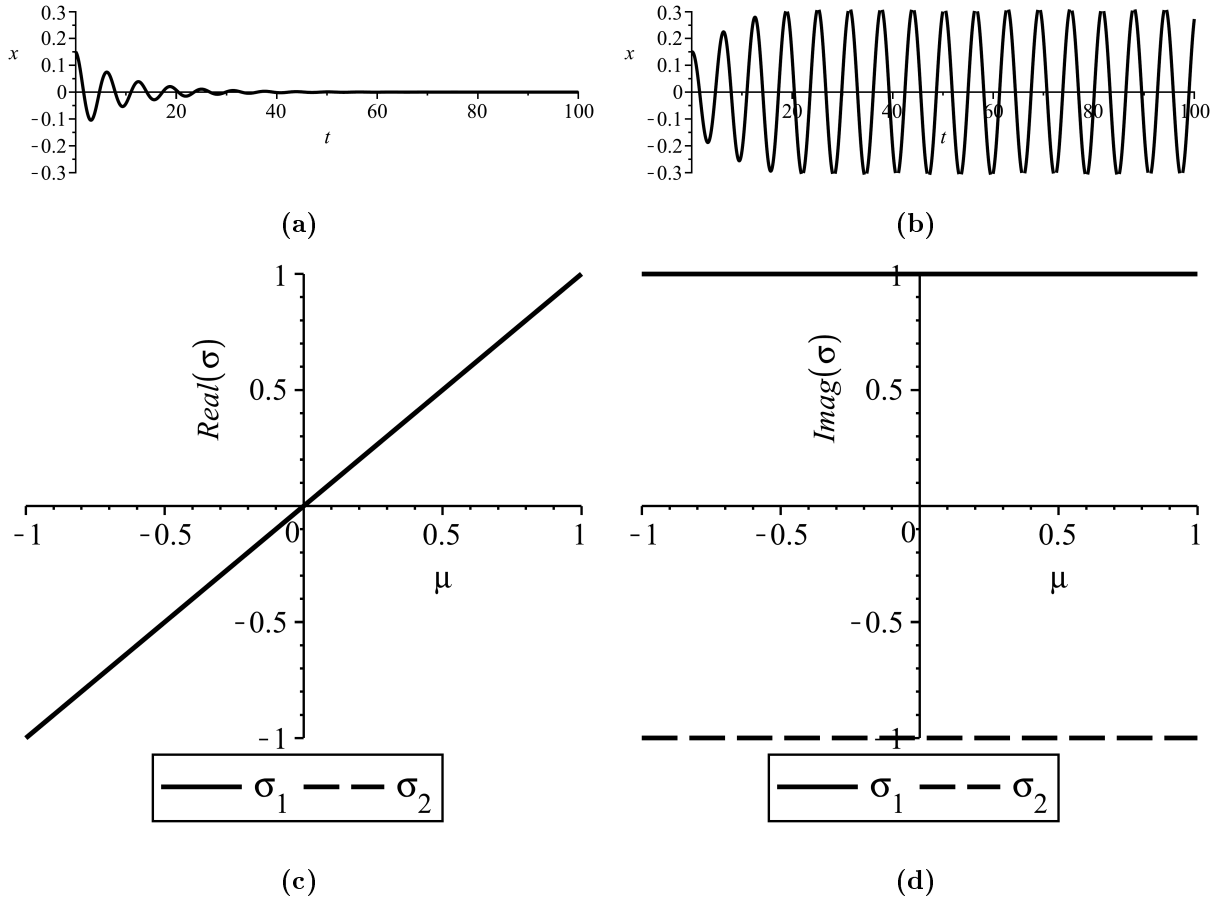


Figure 5.13: A Hopf bifurcation involves the transition of a fixed point from an stable spiral ($\text{Re}\sigma_{1,2} < 0$) to an unstable spiral ($\text{Re}\sigma_{1,2} > 0$). (a) $\mu = -0.1$. (b) $\mu = 0.1$.

We would like to know whether or not the system has a limit cycle, and if so for which values of μ . The only fixed point of this system is $(0, 0)$, and its Jacobian is given by

$$J(0, 0) = \begin{bmatrix} -\mu & -1 \\ 1 & \mu \end{bmatrix}$$

The eigenvalues of this matrix are $\sigma_{1,2} = \mu \pm i$. As the fixed point changes from a stable to an unstable spiral at $\mu = 0$, we conclude that a Hopf bifurcation is taking place at $\mu = 0$. This is confirmed by numerical simulations for $\mu = -0.1$ and $\mu = 0.1$. The transition from a stable to an unstable spiral is clearly discernable from the time series and phase portraits (Figure 5.13(a, b)). The real and imaginary parts of the eigenvalues are plotted as a function of μ in Figs Figure 5.13(c) and Figure 5.13(d), respectively.

For the example under consideration, it can be proved that a limit cycle exists. Expressed in polar coordinates (see Appendix C), the system ((5.19)) becomes:

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= 1 \end{aligned}$$

Hence, the phase angle increases linearly, and the equation for $\dot{r}$ has the normal form of a *supercritical pitchfork bifurcation*. Consequently, we know that the fixed point $r^* = 0$ is stable for

$\mu < 0$ and unstable for $\mu > 0$. Furthermore, a stable fixed point $r^* = \sqrt{\mu}$ exists for $\mu > 0$, which shows that a limit cycle exists for this system.

We conclude this section with some bifurcation jargon for Hopf bifurcations. When a bifurcation take place which involves a transition from a stable spiral to a limit cycle, it is called a *supercritical* Hopf bifurcation. However, it is also possible that the spiral simply becomes unstable without the development of a limit cycle. In this case, the system *loses* a fixed point, and all trajectories within the basin of attraction will suddenly display entirely different behaviour. This is not dissimilar to the catastrophe in the ferromagnetic system discussed in the previous chapter. If the Hopf bifurcation does not generate a limit cycle, the bifurcation is called *subcritical*. One also distinguishes a *degenerate* Hopf-bifurcation, which usually occurs when a nonconservative system becomes conservative at the bifurcation parameter.

5.8.7 Global bifurcations

Global bifurcations are bifurcations that are not confined to a local point in the phase space - hence the word global. Global bifurcations can involve the birth of limit cycles, collisions of limit cycles with fixed points or collisions between limit cycles.

Consider the system

$$\begin{aligned}\dot{x} &= -y + x(\mu + x^2 + y^2 - (x^2 + y^2)^2) \\ \dot{y} &= x + y(\mu + x^2 + y^2 - (x^2 + y^2)^2)\end{aligned}$$

which has a limit cycle which comes into existence at $\mu = -1/4$. Shown in Figure 5.14(a, b) are two phase portraits for a trajectory starting close to the origin. At $\mu = -0.26$, the system can be seen to spiral outwards and eventually escape to infinity. For $\mu = -0.24$ however, the trajectory spirals outward and becomes trapped in the limit cycle. Because this bifurcation involves a change in topology of a finite region of the phase space, the bifurcation is global.

This system is much easier to understand when expressed in polar coordinates (which was of course how it was conceived):

$$\begin{aligned}\dot{r} &= -\mu r - r^3 + r^5 \\ \dot{\theta} &= 1\end{aligned}$$

The equation for the phase angle θ has solution $\theta = \theta(0) + t$. The right hand side of the equation for $\dot{r}$ is plotted in Figure 5.14(c) for three different values of r . Clearly, when $\mu < -1/4$, $\dot{r} > 0$ for all nonzero r and the trajectory will escape to infinity. At $\mu = -1/4$, the function attains a new root in the neighbourhood of $r = 0.7$. This develops into two roots for $\mu > -1/4$.

The fixed points are plotted as a function μ with the help of the function `implicitplot` (Figure 5.14(d)), using the code discussed in section XXX to distinguish between the stable branches (black) and the unstable branches (grey). In this plot it is very clear that two new fixed points, one stable and one unstable, come into existence at $\mu = -0.25$ via a saddle-node bifurcation. As discussed in section XXX, this bifurcation is referred to as a blue-sky bifurcation. It can therefore be said that this system features a “blue sky limit cycle”.

When comparing the phase portraits just before the bifurcation (Figure 5.14(a)) and after (Figure 5.14(b)), it is very clear that even though the bifurcation has not occurred yet at $\mu = -0.26$, the signature of a limit cycle is clearly visible. Indeed, the system nearly gets trapped and takes a long time to escape the vicinity of the limit cycle-to-be. This behaviour is typical for bifurcations. By necessity $f'(r) = 0$ at bifurcation, and dr/dt will therefore become extremely small when μ is close to the bifurcation value.

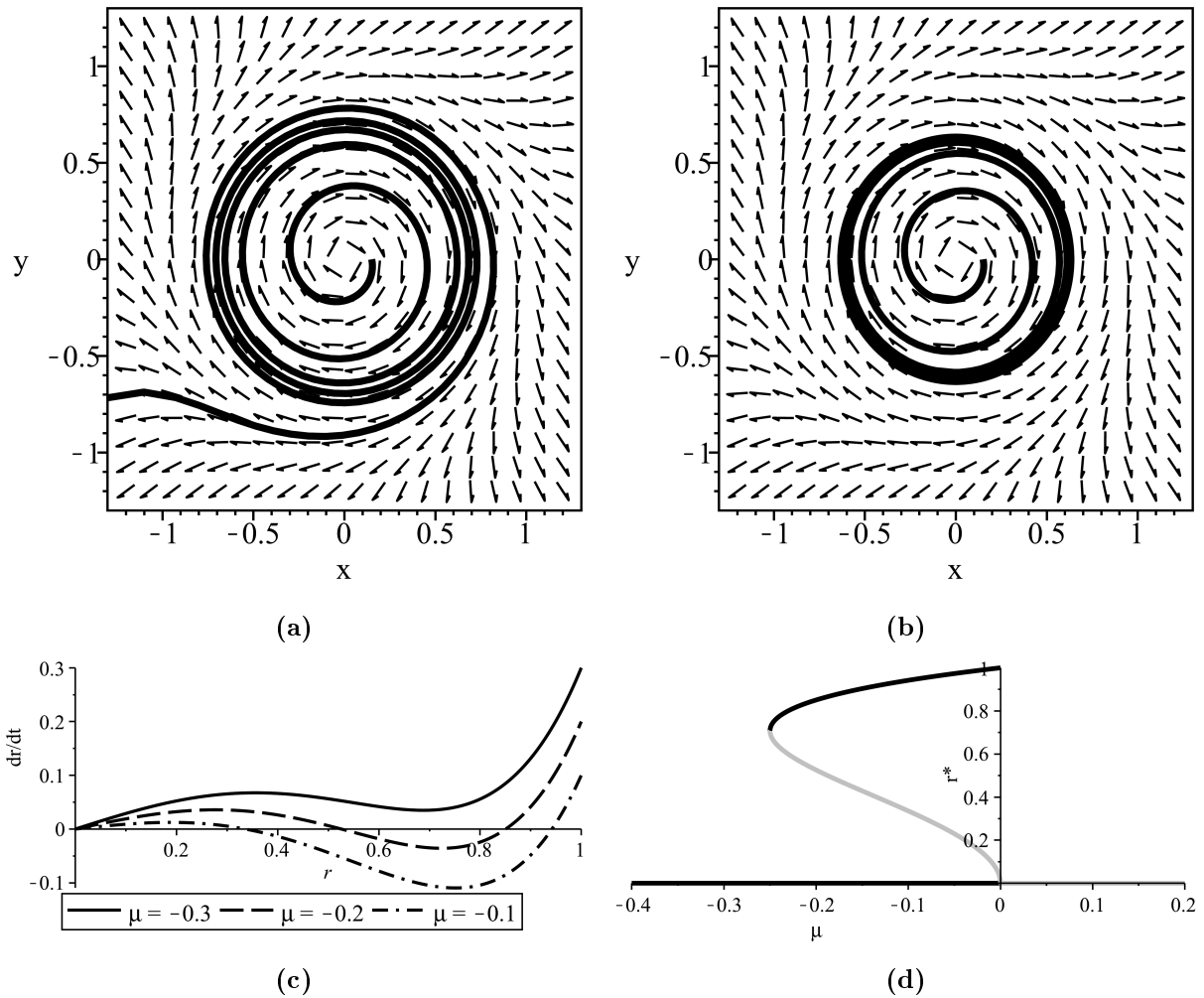


Figure 5.14: A blue sky limit cycle. (a, b) The phase portraits just before and after the bifurcation. (c) dr/dt as a function of r . (d) stability diagram of the fixed points. (a) $\mu = -0.26$. (b) $\mu = -0.24$.

5.9 Exercises

5.9.1 Fixed Points and linearization

Downloads: FixedPointsLinearization.mws

For each of the following systems, find the fixed points and classify them. Next plot the phase portrait with `maple`.

$$\begin{aligned}\dot{x} &= x - y, \quad \dot{y} = x^2 - 4 \\ \dot{x} &= 1 + y - e^{-x}, \quad \dot{y} = x^3 - y \\ \dot{x} &= xy - 1, \quad \dot{y} = x - y^3\end{aligned}$$

5.9.2 Rabbits versus Sheep

Downloads: RabbitsSheep.mws

A rabbit-sheep population with mutual competition can be modelled by the so-called Lotka-Volterra equations

$$\begin{aligned}\dot{N}_1 &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - a_1 N_1 N_2 \\ \dot{N}_2 &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - a_2 N_1 N_2\end{aligned}$$

where it is understood that always $N_1(t), N_2(t) \geq 0$. Furthermore $r_1, r_2 > 0$ and $b_1, b_2 \geq 0$.

a) Show that without loss of generality the equations can be rewritten to

$$\begin{aligned}\dot{x}_1 &= r_1 x_1 (1 - x_1) - b_1 x_1 x_2 \\ \dot{x}_2 &= r_2 x_2 (1 - x_2) - b_2 x_1 x_2\end{aligned}$$

with $x_1(t), x_2(t) \geq 0$.

b) Calculate all fixed points of this system. Give a biological interpretation of each fixed point.

In the sequel we take $r_1 = 3$, and $r_2 = 2$. (What do these value imply from a biological point of view?)

c) Look in the `cookbook.mws` for the command `phaseportrait` to make phase-plane plots. Do this for various values of b_1 and b_2 . In particular, try $b_1 = 0, b_2 = 0$ (no competition); $b_1 = 1, b_2 = 2$; $b_1 = 3, b_2 = 1$; and $b_1 = 4, b_2 = 3$.

d) Calculate with `maple` the linear stability of the fixed points for the case studied in (c). See the `cookbook.mws` for calculating the eigenvalues of a matrix.

e) Take $b_1 = 1$. Find the critical value of b_2 such that the species can still co-exist.

5.9.3 Predator-Prey with Alee effect

Downloads: PredatorPreyAlee.mws

Consider the following *Predator-Prey* system

$$\begin{aligned}\frac{dN}{dt} &= r_1 N \left(\frac{N}{N+B} \right) - \theta_1 NP \\ \frac{dP}{dt} &= r_2 NP - \theta_2 P^2\end{aligned}\tag{5.20}$$

Figure 5.15: Natural convection reactor cooling

where $N(t) \geq 0$ is the number of prey and $P(t) \geq 0$ the number of predators at time t . The parameters $r_1, r_2, \theta_1, \theta_2$ are larger than 0. In the equations the factor $\left(\frac{N}{N+B}\right)$ represents the so-called Allee-effect, which takes into account the effect that larger populations have an advantage over small populations (“mate-finding”, for example, is easier). B is a parameter (≥ 0).

- a) Calculate the fixed points of the system ((5.20)) in terms of the parameters $r_1, r_2, \theta_1, \theta_2, B$.
- b) Show that without loss of generality one can also study the simpler system

$$\begin{aligned}\dot{x} &= \left(\frac{x^2}{x+b}\right) - \frac{xy}{a} \\ \dot{y} &= xy - y^2\end{aligned}\tag{5.21}$$

and express the new parameters a, b in terms of the old parameters $r_1, r_2, \theta_1, \theta_2, B$.

- c) Calculate the fixed points of the system ((5.21)).

Take $b = 1/2$ in the exercises below.

- d) Calculate the stability-matrix J of the non-zero fixed point. Find the characteristic polynomial of J (use the maple command `CharacteristicPolynomial(J,λ)`). Solve the eigenvalues λ from the characteristic polynomial and express them in terms of the parameter a .

- e) Can the system undergo a Hopf-bifurcation? If so, show analytically for which value of a ?

- f) Take $a = 0.75$. Show the phaseportrait of the system, together with the trajectory $\{x(t), y(t)\}$ for initial condition $x(0) = 0.4, y(0) = 0.25$.

- g) Again take $a = 0.75$ and $x(0) = 0.4$. Since the prey population at $t = 0$ is larger than the equilibrium population, biologists have decided to intervene and increase the predator population to $y(0) = 0.25 + \delta$. However, to their utter disbelief they find both species to go rapidly extinct. Study the phaseportrait and trajectories for different δ and find an estimate for the minimal value of δ they must have used.

5.9.4 Nuclear Reactor Cooling

Downloads: NuclearReactorCooling.mws, NuclearReactorCooling.eps

In a nuclear reactor, heat is carried away from the core by convection. Instead of actively forcing this convection by means of a pump, one can also make use of the density difference between the hot, boiling water in the riser and the cooled water in the downcomer to drive the system (see figure Figure 5.15). A large advantage of such an approach is that pump failure and resulting dangerous situations are avoided.

A simplified set of equations describing the change in mass flow rate and temperature in the riser is, respectively:

$$\begin{aligned}\dot{Q} &= aT - bQ \\ \dot{T} &= c - dQT\end{aligned}\tag{5.22}$$

where a accounts for the decrease in average density with higher temperatures, b accounts for friction, c is the normalized power P and d describes the loss of heat due to the outflow, all parameters being *always positive*.

a) Show that, without loss of generality, this set of equations can be written as:

$$\begin{aligned}\dot{x} &= p^2 y - x \\ \dot{y} &= 1 - xy\end{aligned}\tag{5.23}$$

b) Show that in the positive quadrant ($x, y \geq 0$), any fixed point is stable.

c) Find the value(s) for p where the stable fixed point changes in a stable spiral, and show phaseportraits of both cases.

d) Can you find a gradient system to rule out closed orbits? If so, which, and if not, why not?

e) Look at the sign of $\nabla \cdot \dot{\mathbf{x}}$ in the positive quadrant. Can you explain in words why this makes closed orbits in this region impossible? (hint: use Green's theorem):

$$\iint_A \nabla \cdot \dot{\mathbf{x}} \, dA = \oint_S \dot{\mathbf{x}} \cdot \mathbf{n} \, ds$$

6 Chaos in continuous systems

6.1 Strange attractors

The methods discussed in the previous chapter, such as the concept of phase space and the calculation of fixed points and their stability, are generally applicable to systems of arbitrary order that are of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (6.1)$$

For 3-D continuous systems we will adopt the notation $\mathbf{x} = (x, y, z)^T$ and $\mathbf{f} = (f, g, h)^T$. One important distinction between second-order continuous systems and higher-order systems is that the former cannot feature chaotic behavior, as it is topologically impossible for trajectories to cross in phase-space (Poincaré Bendixson theorem). Chaotic behaviour is possible for continuous systems of third order and higher.

As an example, let us return our attention to the celebrated Lorenz equations introduced in chapter 1, which are given by

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz. \end{aligned} \quad (6.2)$$

Figure 6.1 gives an impression of the aperiodic behavior of x for the Lorenz equations (6.2) with $r = 28$, $b = 8/3$ and $\sigma = 10$. The signal flips between positive and negative values, the duration between these reversals different each time. Between reversals, the signal oscillates around a reference value, its amplitude growing with each cycle it completes. When some critical value is exceeded, the signal changes sign, after which the process repeats itself, at least qualitatively.

When all three variables $x(t)$, $y(t)$ and $z(t)$ are plotted in the phase space (Figure 6.2(a)) we get an impression of the *strange attractor* of the Lorenz equations. As for a regular attractor (such

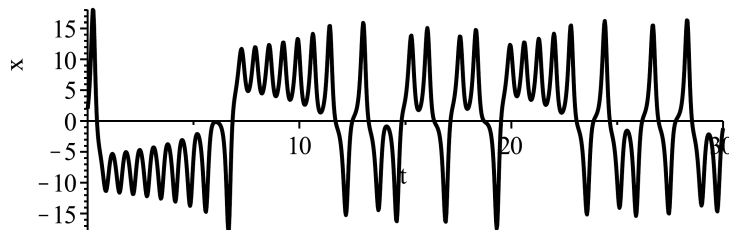


Figure 6.1: A typical timeseries of x showing typical aperiodic behavior.

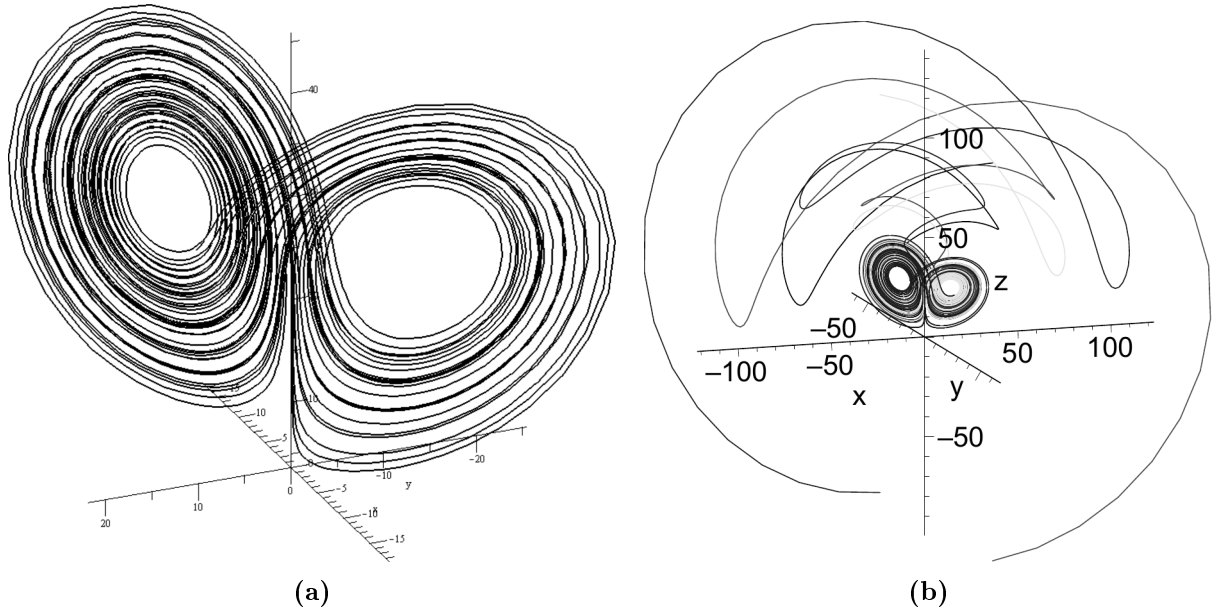


Figure 6.2: The strange attractor of the Lorenz equations. (a) single trajectory $\mathbf{x}(t)$. (b) All trajectories in the basin of attraction end up on the strange attractor.

as a fixed point), the trajectory is caught in a region of the phase space. But in this case, the trajectory meanders erratically back and forth between the two lobes of the attractor. Starting from the center of the right lobe, a trajectory circles outward counterclockwise (from our frame of reference). When it is sufficiently far from the center, it 'falls' onto the other lobe, where it circles outward again, but clockwise. Then, it falls onto the first lobe again but on a different position as before. The trajectory circles outward again, and so on *ad infinitum*.

Trajectories starting far away (Figure 6.2(b)) can be seen to converge upon the strange attractor.

It is time to put the term "strange attractor" on a firmer footing. The formal definition of a strange attractor is quite subtle [Guckenheimer and Holmes, 1983], and we will be satisfied with the following operational definition. The phase space region A is an *attractor* A for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if

- A is invariant under $\mathbf{f}$: any trajectory $\mathbf{x}(t)$ that starts in A cannot escape from A for all times.
- There is a region B , of which A is a subset, for which trajectories $\mathbf{x}(t)$ starting in B are attracted to A in the limit of $t \rightarrow \infty$. This region B is the *basin of attraction* of the attractor A .
- A is minimal: there is no subset of A that satisfies the two conditions above.

When in addition, the trajectories $\mathbf{x}(t)$ on A exhibit a sensitive dependence of initial conditions, it is called a *strange attractor*. The term "strange" originates from the fact that the geometry of strange attractors are often fractal. The terms chaotic attractor or fractal attractor are used as well.

Let us see if the Lorenz equations match these properties. To establish that there is an attractor, we consider a small volume $V(t)$ in phase space and determine the evolution in time of this volume. In this sense one can think of a collection of initial conditions $\mathbf{x}_k(0)$, $k = 1, \dots, \infty$ such that $\forall_k : \mathbf{x}_k(0) \in V(0)$, and imagine $V(t)$ as the volume encompassing the positions $\mathbf{x}_k(t)$ at a

later stage t , i.e. $\forall_k : \mathbf{x}_k(t) \in V(t)$. With $\dot{\mathbf{x}}_k = \mathbf{f}(\mathbf{x}_k)$ defining the flow in phase space, one can derive for the evolution of V

$$\dot{V} = \oint_S \mathbf{n} \cdot \mathbf{f} dS = \int_V \nabla \cdot \mathbf{f} dV \quad (6.3)$$

where the first integral represents a surface integral over the outer surface of V , denoted by S , and $\mathbf{n}$ the normal vector at the surface. The physical interpretation here is that the change in volume is determined by the normal velocity of the bounding surface. The Gauss divergence theorem was used to obtain the volume integral over the divergence of $\mathbf{f}$:

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

When $\nabla \cdot \mathbf{f} = 0$, the system conserves volume and is called *conservative*, the class of Hamiltonian systems (section 5.5.1) being prime examples. When $\nabla \cdot \mathbf{f} < 0$, the volume reduces as a function of time and the system is called *dissipative*. Dissipative systems have attractors but conservative systems do not. However, that both conservative and dissipative systems may display chaotic behaviour.

For the Lorenz equations, the divergence is given by

$$\begin{aligned} \nabla \cdot \mathbf{f} &= \frac{\partial}{\partial x}(\sigma(y - x)) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) \\ &= -\sigma - 1 - b. \end{aligned}$$

As both σ and b are positive, the divergence $\nabla \cdot \mathbf{f} < 0$ and the system is dissipative. As $\nabla \cdot \mathbf{f}$ does not depend on the position, the evolution of $V(t)$ can be determined analytically from (6.3):

$$V(t) = V(0)e^{-(\sigma+1+b)t}$$

This equation confirms that all volumes $V(0)$ in the phase space shrink to zero volume: $V(t) \rightarrow 0$. Here we seem to hit a paradox: if everywhere in the phase space the volume tends to zero, then the resulting geometry must have a dimension lower than three (because it has no volume). But it cannot be two-dimensional either because as we have seen in the previous chapter, there can be no chaos in a flow with only two degrees of freedom (IS POINCARÉ BENDIXSON THEOREM IN C5?). As we will see later, the geometry of the attractor has a fractal dimension, slightly larger than 2, thus yielding an escape from the paradox.

How can one reconcile the shrinking of V to zero on the one hand with the notion of sensitive dependence on initial conditions on the other hand? After all, the latter notion is associated with *diverging* trajectories. This will be investigated numerically, by slightly perturbing a point on the attractor and monitoring the growth of the difference δ . To get an initial condition which is already *on* the attractor, we start with an arbitrary initial boundary condition and integrate the system for some time T . When T is large enough, the trajectory will have converged onto the attractor (see Figure 6.2) and we can take $\mathbf{x}(T)$ as the initial condition. In this way the transient phase is disregarded. With this new initial condition we integrate the system once more and obtain a trajectory $\mathbf{x}_1(t)$ which will be on the attractor. We also study the trajectory $\mathbf{x}_2(t)$ starting from an initial condition that is slightly perturbed: $\mathbf{x}_2(0) = \mathbf{x}_1(0) + \epsilon(0)$, where it is understood that $\|\epsilon\| \ll 1$. In particular we are now interested in the *evolution* of the deviation between the two trajectories, i.e.

$$\epsilon(t) = \mathbf{x}_2(t) - \mathbf{x}_1(t) \quad \delta(t) = \|\epsilon(t)\| = \|\mathbf{x}_2(t) - \mathbf{x}_1(t)\| \quad (6.4)$$

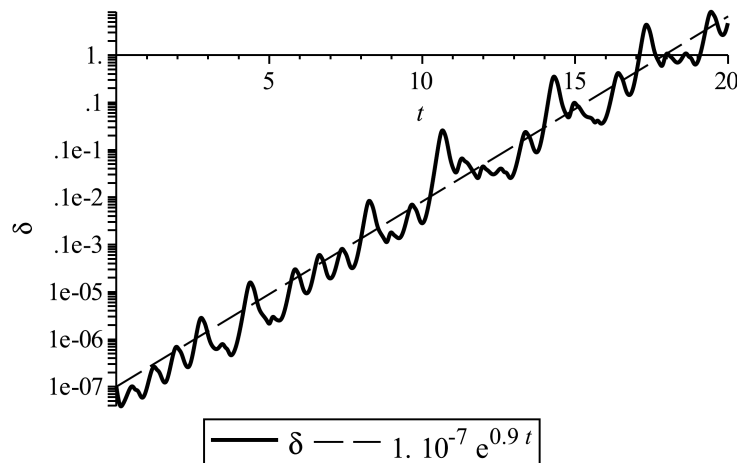


Figure 6.3: The growth of a small perturbation in the initial values. The effective Lyapunov exponent $\Lambda_e \approx 0.9$.



Maple

```
* Define DE, simulate to remove transients, define ic1
...

* Calculate delta
sol_arr := dsolve({DE,ic1},[x(t),y(t),z(t)],type=numeric,maxfun=-1,
  ↪output=timesteps):
soldd:=sol_arr[2,1];
T1 := soldd[1..N, 1]: X1 := soldd[1..N, 2]:
Y1 := soldd[1..N, 3]: Z1 := soldd[1..N, 4]:

epsilon := 1e-7;
ic2 := (x(0)=xx[N]+epsilon,y(0)=yy[N],z(0)=zz[N]);
sol_arr := dsolve({DE,ic2},[x(t),y(t),z(t)],type=numeric,maxfun=-1,
  ↪output=timesteps):
soldd:=sol_arr[2,1];
T2 := soldd[1..N, 1]: X2 := soldd[1..N, 2]:
Y2 := soldd[1..N, 3]: Z2 := soldd[1..N, 4]:
delta := seq([T1[i],sqrt((X1[i]-X2[i])^2+(Y1[i]-Y2[i])^2 + (Z1[i]-Z2[i])^2)],
  ↪i=1..N):
# delta now contains a sequence [t, distance]
```

In Figure 6.3 the evolution of the vectorial distance between the trajectories $\delta(t)$ is shown on a logarithmic scale. Similar to the discrete mappings discussed in section 2.5, we observe that on average the trajectories diverge exponentially

$$\delta(t) = \delta(0)e^{\Lambda_e t} \quad (6.5)$$

with an *effective* Lyapunov exponent $\Lambda_e \simeq 0.9$. As this Lyapunov exponent is positive, there is clearly a sensitive dependence to initial conditions. Note that the distance between the trajectories does not grow monotonically but fluctuates in time. This is because the distance between the trajectories varies along the attractor. At $t = 15$, the distance reaches the size of the attractor and cannot grow any more. At this point the trajectories are no longer correlated to each other. Some

caution needs to be taken with the effective Lyapunov exponent. First, the Lyapunov exponent can depend on the starting position on the attractor. Second, the Lorenz equations have three degrees of freedom, which means that there are three principle directions and therefore also three Lyapunov exponents.

To see the relation between these three Lyapunov exponents and the effective Lyapunov exponent, we first derive for the evolution of the distance between two trajectories

$$\dot{\epsilon} = \mathbf{f}(\mathbf{x}_2) - \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_1 + \epsilon) - \mathbf{f}(\mathbf{x}_1)$$

Assuming $\|\epsilon(t)\| \ll 1$ and ignoring higher order terms, one obtains

$$\dot{\epsilon} = J(\mathbf{x}_1)\epsilon$$

where J is the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}$$

Since $\delta^2 = \epsilon \cdot \epsilon$, we have

$$\frac{1}{2} \frac{d\delta^2}{dt} = \delta \dot{\delta} = \epsilon \cdot \dot{\epsilon} = \epsilon \cdot J(\mathbf{x}_1)\epsilon = \epsilon \cdot J_S(\mathbf{x}_1)\epsilon$$

In the last step, we used that only the symmetric part of the Jacobian $J_S = [J + J^\top]/2$ contributes to the quadratic form. Using (6.5) one can get an instantaneous expression the effective Lyapunov exponent from $\Lambda_e = \dot{\delta}/\delta$, which can be expressed using the equation above in terms of ϵ :

$$\Lambda_e(t) = \frac{\epsilon \cdot J_S \epsilon}{\epsilon \cdot \epsilon} \quad (6.6)$$

So if one has the complete data of the trajectories $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, at any t one can calculate $\epsilon(t) = \mathbf{x}_2(t) - \mathbf{x}_1(t)$ and $J_S(\mathbf{x}_1(t))$, and therefore $\Lambda_e(t)$ using (6.6). A plot of $\Lambda_e(t)$ derived in this way for the Lorenz system is shown in Figure 6.4. A time average of these data points gives $\Lambda_e \simeq 0.9$, consistent with the estimate from $\delta(t)$ in Figure 6.3

(6.6) also provides the link between Λ_e and the three Lyapunov exponents that are present in the system. Indeed, if ϵ is exactly aligned with one of the eigenvectors of J_S , say $\mathbf{e}_1$, then $\Lambda_e = \Lambda_1$, where Λ_1 is the corresponding eigenvalue of J_S . The same holds for the other eigenvectors. So first we note that we can associate the Lyapunov exponents with the eigenvalues J_S ; secondly, if we arrange $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3$ we note from (6.6) that

$$\Lambda_1 \leq \Lambda_e \leq \Lambda_3$$

i.e. the effective Lyapunov exponent is always between the three Lyapunov exponents based on J_S ; thirdly we note that J_S is time dependent as it depends on $\mathbf{x}_1(t)$, so also the eigenvectors depend on time, and also the Lyapunov exponents $\Lambda_1(t)$, $\Lambda_2(t)$, $\Lambda_3(t)$, see Figure 6.5.

Figure 6.5 shows all Lyapunov exponents as a function of time. Apart from the strong fluctuations in time one notices that generally one Lyapunov exponent is negative, one positive and one near zero. This appears to be rather typical for chaotic systems. The negative exponent causes the contraction of trajectories towards the attractor, whereas the positive exponent causes the divergence along the attractor and hence the sensitivity to initial conditions. One also notes the

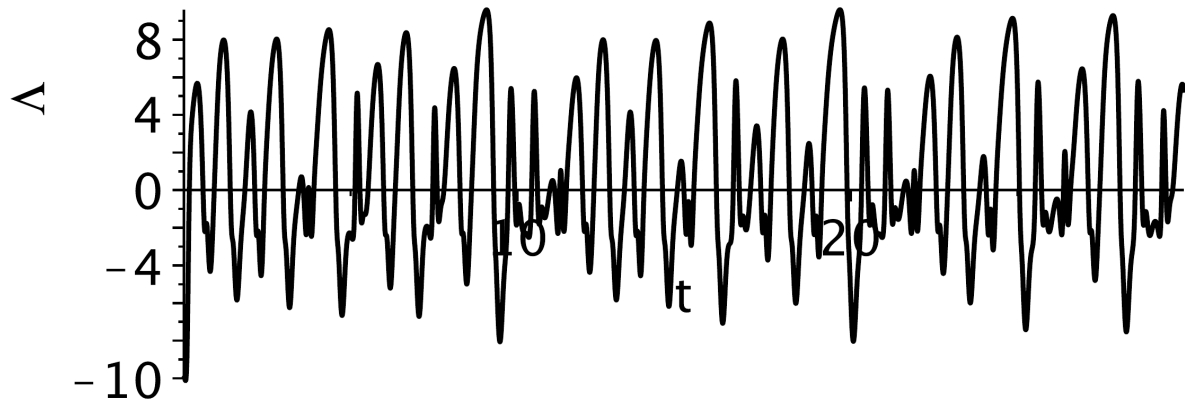


Figure 6.4: Instantaneous (effective) Lyapunov exponent Λ_e determined for the Lorenz system from (6.6).

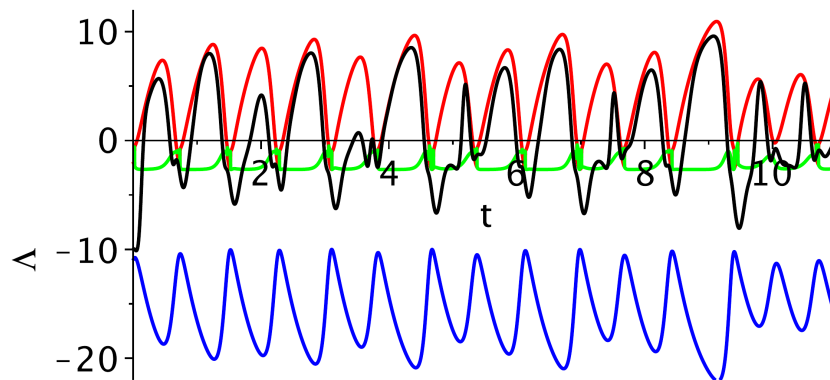


Figure 6.5: The three Lyapunov exponents of the Lorenz system as well as the effective Lyapunov exponent Λ_e (black line). Bottom: Cumulative average of the Lyapunov exponents in time, showing the convergence to $\Lambda_e \simeq 0.9$.

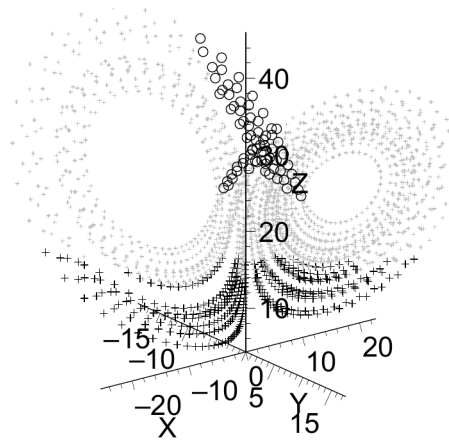


Figure 6.6: Effective Lyapunov exponent Λ_e on the attractor. Circles denote $\Lambda_e > 6$, and crosses $\Lambda_e < -6$

fact that Λ_e is always between the smallest and the largest Lyapunov exponent, but also that the effective Lyapunov exponent is by no means equal to the average $(\Lambda_1 + \Lambda_2 + \Lambda_3)/3$. Finally, it is interesting to study whether there are specific regions on the attractor that can be associated with strong divergence ($\Lambda_e \gg 0$), or strong convergence events ($\Lambda_e \ll 0$). This is indeed the case, as shown in Figure 6.6.

The following maple program performs the calculation of the three Lyapunov exponents and the effective Lyapunov exponent such as shown in Figure 6.5.



Maple

```

restart; with(plots): with(DEtools): with(LinearAlgebra):
Digits := 20;
f1 := (x,y,z) -> sigma * (y - x);
f2 := (x,y,z) -> -x * z + r * x - y;
f3 := (x,y,z) -> x * y - b * z;
DE1 := diff(x(t),t) = f1(x(t),y(t),z(t));
DE2 := diff(y(t),t) = f2(x(t),y(t),z(t));
DE3 := diff(z(t),t) = f3(x(t),y(t),z(t));
DE := DE1, DE2, DE3;
r:=28; b:=8/3; sigma:=10;
ic0 := x(0)=2,y(0)=5,z(0)=5;
timesteps := Array([10]); # integrate to t=10 to remove transient
sol_arr := dsolve({DE,ic0},[x(t),y(t),z(t)], type=numeric,maxfun=-1,
  ↪output=timesteps):
soldd:=sol_arr[2,1]:
X := soldd[1, 2];
Y := soldd[1, 3];
Z := soldd[1, 4];
x0 := X; y0 := Y; z0 := Z;
ic1 := (x(0)=x0,y(0)=y0,z(0)=z0); # initial condition on the attaractor
N:=6000; dt := 0.005;
timesteps := Array(1..N, i -> (i-1)*dt):
sol_arr := dsolve({DE,ic1},[x(t),y(t),z(t)],type=numeric,maxfun=-1,
  ↪output=timesteps):
soldd:=sol_arr[2,1]:
T1 := soldd[1..N, 1]:
X1 := soldd[1..N, 2]:
Y1 := soldd[1..N, 3]:
Z1 := soldd[1..N, 4]:
epsilon := 1e-12;
ic2 := (x(0)=x0+epsilon,y(0)=y0,z(0)=z0);
sol_arr := dsolve({DE,ic2},[x(t),y(t),z(t)],type=numeric,maxfun=-1,
  ↪output=timesteps):
soldd:=sol_arr[2,1]:
T2 := soldd[1..N, 1]:
X2 := soldd[1..N, 2]:
Y2 := soldd[1..N, 3]:
Z2 := soldd[1..N, 4]:
VectorCalculus[Jacobian]([f1(x,y,z), f2(x,y,z),f3(x,y,z)], [x,y,z]);
J := unapply(%, [x,y,z]):
(J(x,y,z) + Transpose(J(x,y,z)))/2;
Js := unapply(%, [x,y,z]): # Symmetric part of the Jacobian
l1 := Array(1..N): l2 := Array(1..N): l3 := Array(1..N): le := Array(1..N):
for i from 1 to N do
  ev := Eigenvalues(Js(X1[i],Y1[i],Z1[i]));
  l1[i] := Re(ev[1]); l2[i] := Re(ev[2]); l3[i] := Re(ev[3]);
  eps := Vector([X2[i]-X1[i], Y2[i]-Y1[i], Z2[i]-Z1[i]]);
  epsdot := J(X1[i], Y1[i], Z1[i]) . eps;
  le[i] := (epsdot . eps)/(eps . eps);
end do:
pl1 := plot([seq([T1[i],l1[i]],i=1..N)],color=blue,thickness=2):
pl2 := plot([seq([T1[i],l2[i]],i=1..N)],color=red,thickness=2):
pl3 := plot([seq([T1[i],l3[i]],i=1..N)],color=green,thickness=2):
ple := plot([seq([T1[i],le[i]],i=1..N)],color=black,thickness=2):
display([pl1,pl2,pl3,ple],labels=["t", 'Lambda']);

```

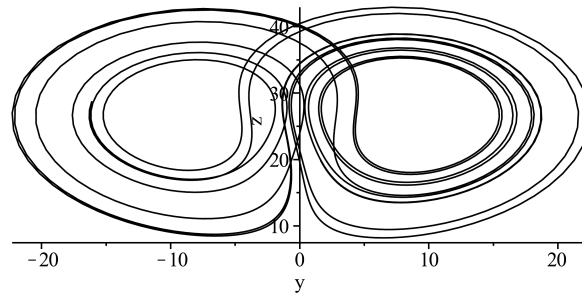


Figure 6.7: Phase space projection to the $y - z$ plane. Note the apparent relation between succeeding maxima of z .

6.2 Reducing complexity

One of the challenges of analyzing higher order systems is that the dynamics are more complex and harder to visualize than for 1-D and 2-D systems. The complex system dynamics be made more tractable in several ways, for example by

- analysing projections of the phase space trajectory;
- recording the maxima of one of the variables;
- recording the time and location where the trajectory intersects a surface in phase space. This is called a Poincaré section.
- recording the phase space periodically. This is often used in forced systems, and is also called a stroboscopic Poincaré section.

All but the first item comprise a reduction from a continuous system to a discrete system. Consequently, many of the concepts we have covered in earlier chapters, such as the return plots and the routes to chaos will be encountered in the following sections.

6.2.1 Phase space projection and sampling of maxima

We shall use the Lorenz equations as an example to show how projections and the sampling of maxima can be used to analyse continuous systems. One may ask oneself the question whether the data in Figure 6.1 and Figure 6.2 is truly nonperiodic or simply has a very long return period. In his original paper, Lorenz [1963] sampled the maxima in Z to show that stable limit cycles do not exist (and hence the flow is nonperiodic), as will be elaborated below.

Consider an $y - z$ projection of the strange attractor timeseries of the z -variable (Figure 6.7). Focusing on the trajectories, it seems that there is a relation between one maximum in z and the next. The trajectories spiral outward until some critical value is reached, after which it "falls" on the other lobe. The higher the trajectory is before the transition, the more it seems to end up in the center of the other lobe. Apparently, the maxima of z are to some extent characteristic for the system. Lorenz collected the successive maxima z_n by integrating the system for a sufficiently long period and obtained a sequence of successive maxima z_n (Figure 6.8). In maple, this can be achieved by

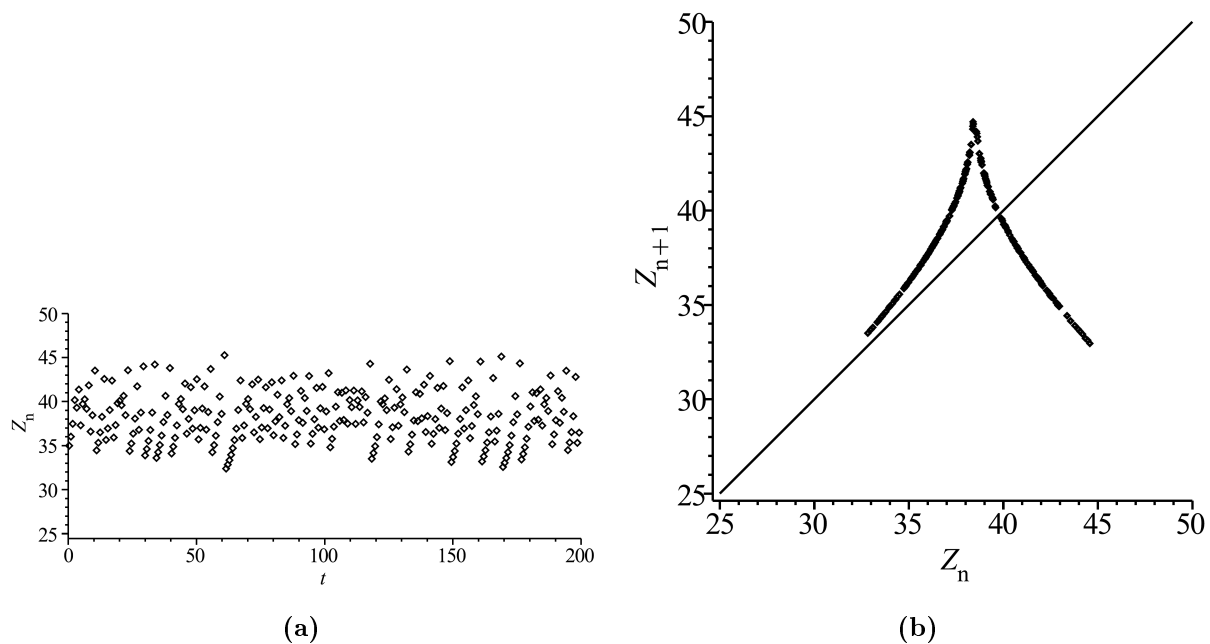


Figure 6.8: Construction of the Lorenz-map (b) of the Lorenz system, obtained by making a return plot of the data in Figure 6.8. (a) The local maxima z_n obtained from a time series $z(t)$ of the Lorenz system. (b) The Lorenz map.



Maple

* T,X,Y,Z are the Arrays containing t,x,y and z-data respectively.
 * N is the number of elements

```
n := 0;
for j from 2 to N-1 do
  if( Z[j-1] < Z[j] and Z[j] >= Z[j+1] ) then # found a maximum
    n := n + 1;
    Tmx[n] := T[j];
    Zmx[n] := Z[j];
  end if;
end do;
```

Note that the transition from a 3-D continuous to a 1-D discrete system is a formidable reduction in complexity!

With a discrete series z_n , plotted in Figure 6.8, we have entered familiar territory (chapter 2), and the natural next step is to study whether the relation between z_n and z_{n+1} can be described by a one-dimensional discrete mapping $z_{n+1} = f(z_n)$. This can be done by looking at the return plot of z_n , shown in Figure 6.8. This plot defines the *Lorenz map* of the continuous Lorenz system (6.2). Indeed, the sequence appears to collapse onto a practically symmetric curve with a cusp in the center. To a large extent the return-plot reminisces the tent map (2.29), but closer inspection reveals that the curve in Figure 6.8 has a finite thickness and formally cannot be described by a one-dimensional mapping. However, the thickness is so small that for the sake of argument the relation between z_{n+1} and z_n will still be treated as a one-dimensional map.

So how does the Lorenz map help in ruling out stable limit cycles? Sampling the maxima in $z(t)$

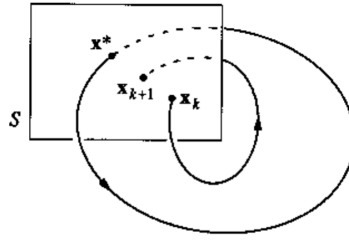


Figure 6.9: A Poincaré section samples the intersections of a trajectory in phase space with the surface S . [Strogatz1994].

of a limit cycle would result in a *finite* sequence of order m , $z_1, z_2, \dots, z_m$ after which the sequence repeats itself. Since the sequence is generated by the Lorenz map $z_{n+1} = f(z_n)$, we can use (2.21) to determine the stability $\sigma^{(m)}$ of this periodic solution:

$$\sigma^{(m)} = f'(z_1)(f'(z_2)) \dots f'(z_m). \quad (6.7)$$

As discussed in section 2.3.2, the criterion for stability is given by $|\sigma^{(m)}| < 1$. Now the remarkable observation to be made in Figure 6.8 is that at each z , the slope $|f'(z)|$ is larger than one. By (6.7) it is therefore inevitable that $|\sigma^{(m)}| > 1$ for all possible sequences, and hence there are no stable limit cycles possible. In this way Lorenz showed that the Lorenz system indeed produces a deterministic *non-periodic* flow in phase space¹. In the early sixties, this concept was truly groundbreaking. Associated with the term deterministic were systems, whose long term behavior was either to settle in a fixed point or end up in a limit cycle. Nonperiodic solutions on the other hand, were associated with random forcings which prevented the system from settling in a limit cycle or fixed point. But here was a deterministic system of ordinary differential equations with only three degrees of freedom which produced a flow that *never* repeats itself!

6.2.2 Poincaré sections

The idea behind a Poincaré section is to reduce the complexity of an N -dimensional continuous dynamical system to a $N - 1$ -dimensional *discrete* map by inserting a *surface of section* S into the phase space and sampling the intersections of a trajectory $\mathbf{x}(t)$ with S . This is particularly useful for oscillating systems, i.e. where the trajectory orbits in a limit cycle or has chaotic behavior. In Figure 6.9, a typical trajectory is sketched which starts on the surface of section S at point $\mathbf{x}_k$. The trajectory $\mathbf{x}(t)$ is governed by the flow $\mathbf{f}(\mathbf{x}(t))$, and after a certain amount of time, the trajectory intersects S again at $\mathbf{x}_{k+1}$, and so on.

The mapping $\mathbf{x}_{k+1} = P(\mathbf{x}_k)$ is called the *Poincaré map*. The surface of section does not need to be a plane, but can be curved if convenient for the flow under consideration. The main requirement to choose the surface of section is that the trajectories must intersect it: the surface should not be parallel to the flow.

A simple limit cycle will correspond to a fixed point $\mathbf{x}^* = P(\mathbf{x}^*)$ on the surface of section S (Figure 6.9). If the limit cycle is more complex than shown, i.e. when it requires M intersections to return back to its original position, then the discrete mapping has period- M behavior. For

¹ Note that the method to rule out stable limit cycles does not constitute a rigorous proof: the Lorenz map is obtained by long term numerical integration and is not constructed analytically.

chaotic behavior, the Poincaré map will never return to the same place, and a low dimensional version of the attractor will emerge (for dissipative systems).

As dynamical systems are often nonlinear, a Poincaré section can in general only be obtained by numerically integrating the system and storing the intersections with the surface of section. Therefore, the Poincaré map P is in general not known explicitly. To know P explicitly, one would need the solution of the dynamical system, which only occurs in rare circumstances. However, it is quite insightful to how a Poincaré map is constructed from a dynamical system.



Example

Consider the two-dimensional system

$$\begin{aligned}\dot{x} &= -y + ax(1 - x^2 - y^2) \\ \dot{y} &= x + ay(1 - x^2 - y^2),\end{aligned}$$

where a is a positive parameter. This system contains a limit cycle, as shown in [Figure 6.10a](#) for $a = 0.2$. We define our "surface" of section as the line $x > 0$, $y = 0$. As the system is two-dimensional, the Poincaré section will result in a discrete one-dimensional map.

The system above can be solved analytically by converting it to polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$, yielding

$$\begin{aligned}\dot{r} &= ar(1 - r^2), \\ \dot{\theta} &= 1.\end{aligned}$$

which has a solution

$$\begin{aligned}r(t) &= \left(1 - \frac{r_0^2 - 1}{r_0^2} e^{-2at}\right)^{-1/2} \\ \theta(t) &= t + \theta_0.\end{aligned}$$

Here $r_0 = r(0)$ and $\theta_0 = \theta(0)$ are the initial conditions.

The surface of section $x > 0$, $y = 0$ transforms to $\theta = 0 \pmod{2\pi}$, so that the Poincaré map $r_{n+1} = P(r_n)$ becomes

$$P(r) = \left(1 - \frac{r^2 - 1}{r^2} e^{-4a\pi}\right)^{-1/2},$$

where we used that $\theta_0 = 0$. This mapping is shown in [Figure 6.10b](#) for $a = 0.2$. Clearly, $r^* = 1$ is a stable fixed point of this mapping, which confirms the convergence to a limit cycle.

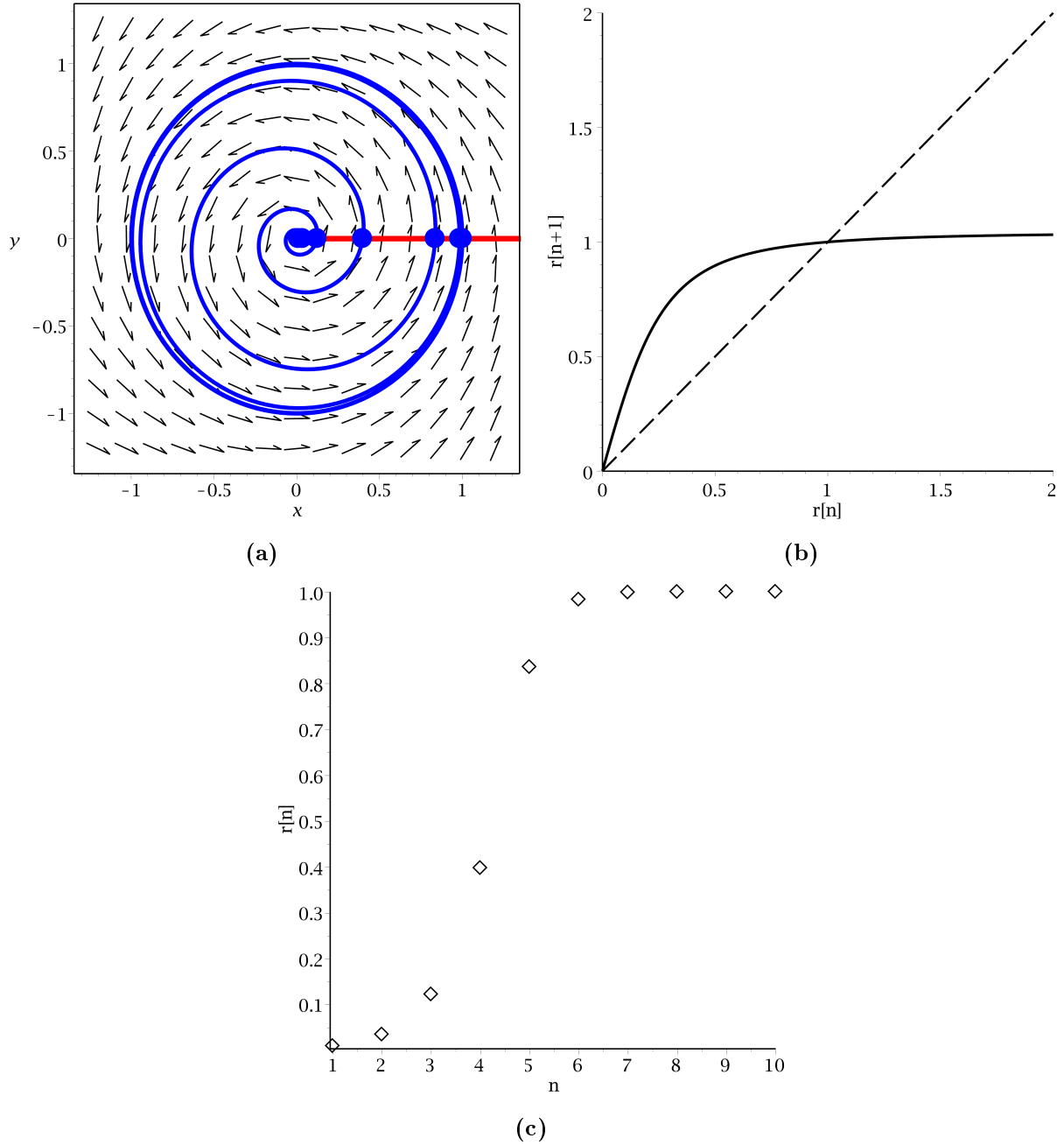


Figure 6.10: Phaseportrait and the mapping for the Poincaré section. (a) Phase portrait with surface of section (red line) and the intersections (solid circles). (b) The Poincaré map. (c) Some iterations of the Poincaré map, showing the approach towards the fixed point in r .

**Example**

In most circumstances, the Poincaré map will need to be constructed from the numerical solution. As an example, consider again the Lorenz system and a surface of section given by $y = 0$. The following code fragment extracts the relevant data and plots the section (assumes that the solution is sampled with high time resolution and is stored in the arrays `xx`, `yy` and `zz`):

```
pcx := NULL: pcz := NULL:
for i from N/4 to N-1 do
  if (yy[i] < 0 and yy[i+1] > 0 or yy[i] > 0 and yy[i+1] < 0) then
    pcx := pcx, xx[i]; pcz := pcz, zz[i];
  end if;
end do;
plot([seq([pcx[i], pcz[i]], i=1..nops([pcx]))], style=point,
      symbol=diamond);
```

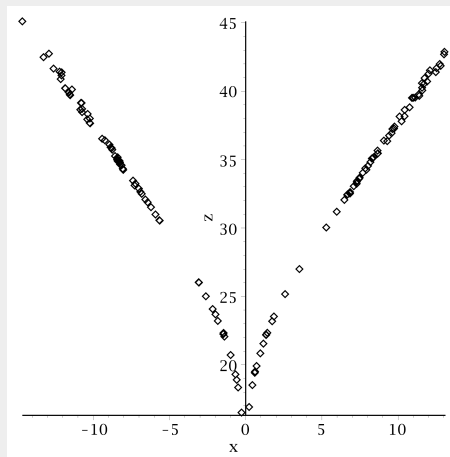


Figure 6.11: Poincaré section at $y = 0$ for the Lorenz system.

The result is shown in Figure 6.11. As can be seen, the discrete mapping results in a (nearly) one-dimensional curve, which has strong similarities with the Lorenz map (Figure 6.8). Note that the integration has only been carried out for a limited amount of time – longer integration would have resulted more data points on the map and therefore less holes.

6.3 Routes to chaos

There is no universality in the route by which continuous systems become chaotic. Indeed, the route to chaos depends on the specifics of the system under consideration. Some systems will feature a classical route to chaos through period doubling, whilst others become chaotic instantaneously. Several other routes to chaos have been identified over the years. These are beyond the scope of this text; see e.g. [Schuster, 1995] for a review.

A classical period-doubling route can be observed in the Rössler system [Rossler, 1976]:

$$\begin{aligned}\dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c),\end{aligned}$$

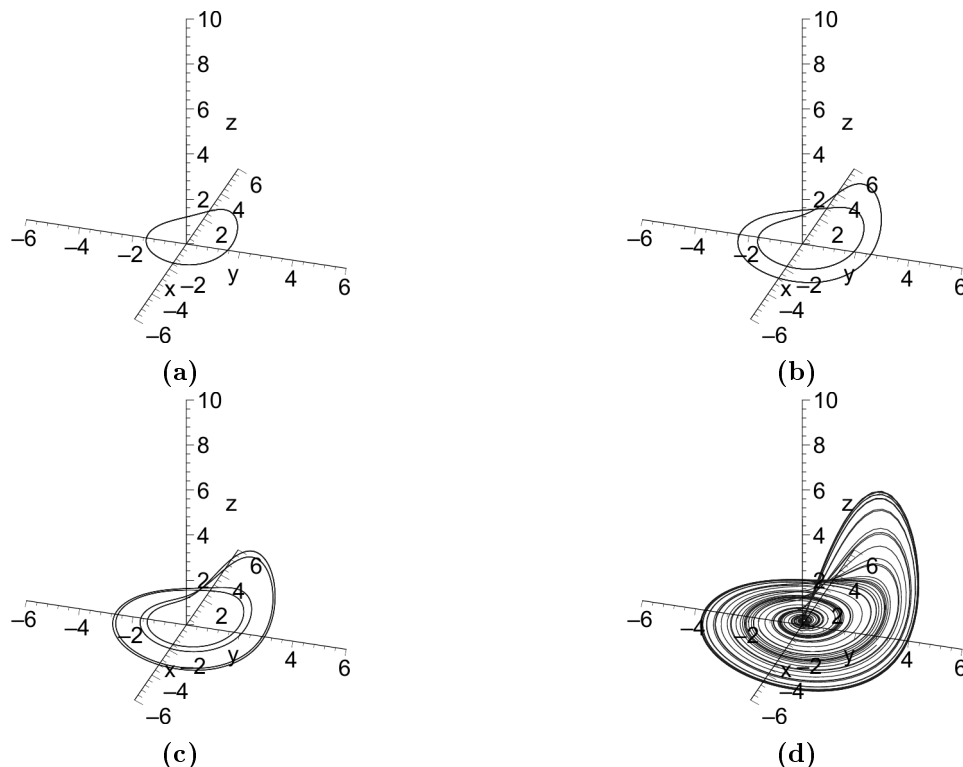


Figure 6.12: The period doubling route to chaos of the Rössler system. (a) $c = 3$: Period 1. (b) $c = 3.4$: Period 2. (c) $c = 3.55$: Period 4. (d) $c = 4.0$: Chaos.

with parameters a , b and c . Note that only the z -equation contains a nonlinearity. In the absence of z , the first two equations are just those of a harmonic oscillator. As the parameter a is positive there is negative damping, so that the typical behavior when $z \approx 0$ will be that of an unstable spiral. The attractor is shown in Figure 6.12(d) for $a = 0.432$, $b = 2$ and $c = 4$. Following a trajectories starting at the center at $z = 0$, the trajectory spirals outward. When x becomes sufficiently large (after one or more cycles), the z equation becomes active and the trajectory shoots up. Due to the negative feedback in the x -equation it is then reinjected near the center of the spiral. Trajectories are reinjected further to the inside of the spiral when they were further on the outside, generating a surface similar to a Möbius strip. Note that the attractor is a topologically very complex object, as the outward spiralling trajectories are weaved through the reinjected trajectories.

As for the Lorenz map, we sample the maxima of x in order to reduce the complexity of the data. In Figure 6.12, the behavior of the system is shown for four different values of c . For $c = 3$ (Figure 6.12a), the system settles into a limit cycle, and subsequent maxima in x will be identical. This situation resembles that of a fixed point (period-1) solution for a one-dimensional map. For $c = 3.4$ (Figure 6.12b), the system settles into a limit cycle which has become more complex: the trajectory performs two cycles before returning to the same location. This is reflected in the sequence of the x -maxima, which shows a period-2 solution. A period-4 limit cycle is shown in Figure 6.12c and the chaotic timeseries in Figure 6.12d. In fact, the Rössler system has a full period doubling route to chaos, which is shown in Figure 6.14a. Clearly, there is a strong similarity with the bifurcation diagram of the logistic map, Figure 2.11.

A clue for the period-doubling behavior could have been inferred from the return plot of z_n in a fully chaotic state ($c = 4$) (Figure 6.13). The return plot is very similar to the logistic map, and

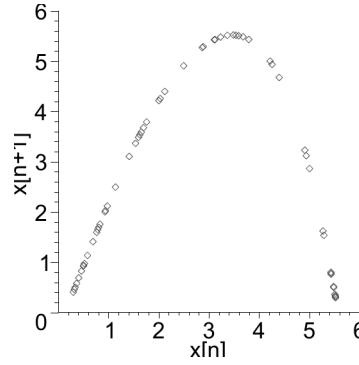


Figure 6.13: Return map based on maxima in x for the Rössler system at $c = 4$

has both requirements for Feigenbaum's universality (section 2.8): 1) a unimodal map; and 2) a differentiable maximum. From the Lorenz map (Figure 6.8) one might anticipate that the Lorenz system will behave rather differently than the Rössler system, as the cusp implies that the map is not differentiable at its maximum. Shown in Figure 6.14b is the bifurcation diagram in terms of r for the Lorenz equations. The other two parameters are $\sigma = 10$ and $b = 8/3$. As can be seen, the system is stable up to a certain value for r , after which it bursts into chaotic behaviour without a single trace of period doubling.

The critical value of r for the transition to chaos for the Lorenz equations may be obtained using the linear stability analysis for the fixed points outlined in the previous chapter. The fixed points are given by $x_1^* = (0, 0, 0)$, $x_2^* = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $x_3^* = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$. The origin $(0, 0, 0)$ is a stable fixed point for $0 < r < 1$. At $r = 1$ a pitchfork bifurcation occurs, where two nonzero fixed points become stable. The stability range of x_2^* and x_3^* is

$$1 < r < \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}.$$

at $r_c = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$, x_2^* and x_3^* undergo a Hopf bifurcation. Recall that the Hopf bifurcation represents a transition from a stable spiral (negative real part of eigenvalue) to an unstable spiral (positive real part of eigenvalue). In this case there is no stabilizing flow away from the fixed point, which means that no limit cycle can form; this bifurcation thence marks an immediate transition to chaos².

6.4 Forced systems

The Poincaré-Bendixson theorem states that 2-D continuous systems cannot have chaotic behaviour. However, this only pertains to unforced (autonomous) systems – if 2-D systems are forced they may become chaotic because a non-autonomous 2-D system is equivalent to an autonomous system of one degree higher. Consider for example a forced oscillator governed by

$$m\ddot{x} = F + A \sin \omega t$$

where x is the position, m is the mass, $F(x, \dot{x})$ is the force, and A and ω are the forcing amplitude and frequency, respectively. The equation above is a non-autonomous second-order differential equation that can be converted to an autonomous system by introducing the variables $y = \dot{x}$ and

² for an elaborate and quantitative treatment of the Lorenz equations, see [Sparrow, 1982]

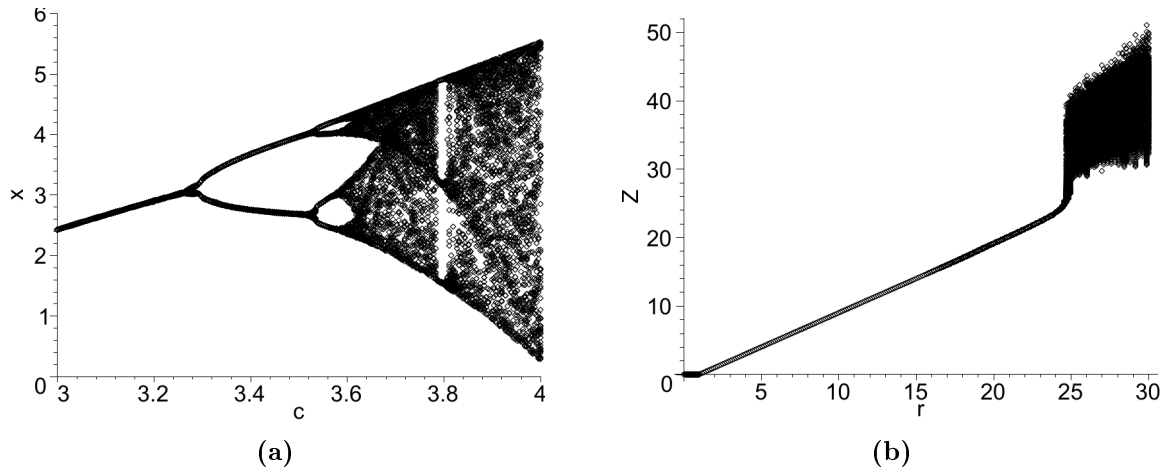


Figure 6.14: Bifurcation diagram of the Rössler and Lorenz equations based on maxima in x and z , respectively. (a) Rössler system. (b) Lorenz system.

$z = \omega t$. The result is

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= F/m + (A/m) \sin z \\ \dot{z} &= \omega\end{aligned}$$

Even though the equation for z is trivial to solve, this result is significant because it incorporated non-autonomous systems into the geometrical interpretation that the phase space provides.

A commonly used concept to reduce the complexity of forced systems is to sample the state vector $\mathbf{x}(t)$ at time fixed intervals $2\pi/\omega$. By sampling at the same frequency as the forcing, information is provided about whether the system displays forced periodic behavior or has more complex dynamics. This is commonly referred to as a *stroboscopic Poincaré section*.

**Example**

Consider a damped and forced version of the double-well oscillator discussed in section 5.5.1 governed by

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^3 - fy + A \cos(z) \\ \dot{z} &= \omega\end{aligned}\tag{6.8}$$

Here, f is a damping coefficient and ω and A are the forcing frequency ω and amplitude A , respectively. Physically, this system can represent a slender steel beam with a damping coefficient f that is clamped in a rigid framework, with two permanent magnets at the base pulling the beam in opposite directions (Figure 6.15). As we have seen in section 5.5.1, in the absence of friction and forcing ($f = 0$ and $A = 0$) this system has a potential $V = x^4/4 - x^2/2 + y^2/2$ (Figure 5.9 or Figure 6.16(a)) that has two minima representing the stable equilibrium solutions in which the beam is buckled to one side; the problem earns its name due to the two wells that form as a result.

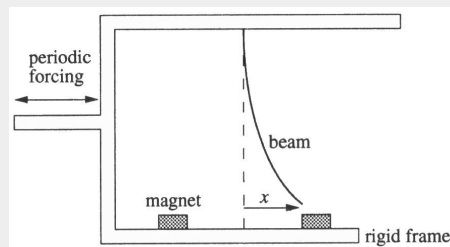


Figure 6.15: Sketch of the mechanical system [Strogatz1994].

In the absence of the forcing, the system has three fixed points, $\mathbf{x}^* \in \{(-1, 0), (0, 0), (1, 0)\}$. The fixed point at the origin is a saddle node and the other two are stable fixed points. For low values of A , the damping will exceed the forcing and the system will end up in one of the wells in a limit cycle. If the forcing amplitude A exceeds a certain value, the system is perturbed so strongly that it is no longer confined to one well only and can move between the two wells. An example is shown in Figure 6.16(a) for parameter values $\omega = 1$, $f = 0.15$, $A = 0.3$, and initial conditions $x(0) = 0$, $y(0) = 0$ and $z(0) = 0$. Apart from the trajectories, isolines of the potential V and the three fixed points are shown. The code used to produce the figure is given below.

```
restart; with(plots):
f1 := (x, y, z) -> y;
f2 := (x, y, z) -> -f * y + x - x^3 + A*cos(z);
f3 := (x, y, z) -> omega;
de1 := diff(x(t), t) = f1(x(t), y(t), z(t));
de2 := diff(y(t), t) = f2(x(t), y(t), z(t));
de3 := diff(z(t), t) = f3(x(t), y(t), z(t));
sys := de1, de2, de3;
# Plot total energy of system and fixed points
H := (x,y) -> x^4/4 - x^2/2 + y^2/2;
p1 := contourplot(H(x,y), x=-2..2, y=-2..2, grid=[70, 70]):
p2 := plot([-1, 0, 1], [0, 0, 0], symbol=circle, style=point):
# Calculate double-well oscillator dynamics and plot the results
f := 0.15; omega := 1; A := 0.3;
ic := x(0) = 0, y(0) = 0, z(0) = 0;
tend := 1000;
sol := dsolve({sys, ic}, type=numeric, maxfun=-1, output=listprocedure):
p3 := odeplot(sol, [x(t), y(t)], 800..tend, numpoints=tend*50)
display([p1, p3, p2]);
```

6.5 Chaos in conservative systems

As a final topic of this book we consider the three-body problem that was briefly touched upon in chapter 1. The three-body problem is a canonical problem in celestial mechanics that was pivotal in the realisation that deterministic systems are not always predictable. Whilst the two-body problem is relatively straightforward to solve, as was done by Kepler and formalised by Newton in the 17th century, the solution to the three-body problem remained elusive for several centuries. Newton tried to solve the dynamics of the sun-earth-moon system and remarked about this problem that "...his head never ached but with his studies on the moon"³. It was a strong-held belief that solving the three-body problem was merely a matter of time but despite important contributions made by heavy-weights like Euler and Lagrange the problem was not solved. It was not until the early 20th century that Henri Poincaré showed unambiguously that there are fundamental issues with the integrability of the three-body system, and thus its predictability.

NEEDS DIAGRAM

Poincaré's work mostly focussed on the restricted three-body problem, which is a simplified version of the three-body problem in which 1) one body is assumed much smaller than the other two, e.g. a sun-moon-satellite problem; 2) the two heavy bodies rotate in the same plane; and 3) the two heavy objects rotate around each other in a circular motion. Even in this very simplified setting the system governing the motion of the lightest body (the satellite) is governed by a system of four coupled ordinary differential equations:

$$\begin{aligned}\dot{X} &= U \\ \dot{Y} &= V \\ \dot{U} &= Gm_1[X_1 - X]/d_1^3 + Gm_2[X_2 - X]/d_2^3 \\ \dot{V} &= Gm_1[Y_1 - Y]/d_1^3 + Gm_2[Y_2 - Y]/d_2^3\end{aligned}\tag{6.9}$$

where $(X(t), Y(t))$ and $U = \dot{X}, V = \dot{Y}$ are the satellite coordinates and velocities, respectively. By convention, we assume $m_1 > m_2$. The variables d_1 and d_2 are distances to the heavy bodies, given by

$$d_1 = \sqrt{(X_1 - X)^2 + (Y_1 - Y)^2} \quad d_2 = \sqrt{(X_2 - X)^2 + (Y_2 - Y)^2}\tag{6.10}$$

and earth- and moon-positions

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = -\frac{m_2 R}{m_1 + m_2} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \quad \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \frac{m_1 R}{m_1 + m_2} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}\tag{6.11}$$

This is a problem without friction and the system is thus conservative. The most convenient way to see this is to consider the system in terms of a coordinate system that co-rotates with the two bodies; the two heavy bodies are static in this reference frame, which considerably simplifies the analysis. The drawback of the transformation is that the rotating coordinate system introduces pseudo-forces, i.e. centripetal and Coriolis forces. For a derivation, we refer to [Tél and Gruiz, 2006]. The result is, after making the system dimensionless using the distance between the two

³ <http://www.newtonproject.sussex.ac.uk/catalogue/record/THEM00169>

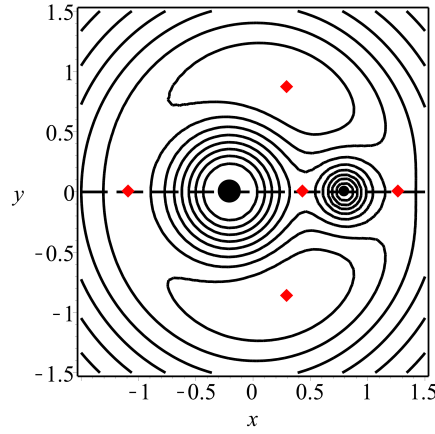


Figure 6.17: The potential V for the restricted three-body system at $\mu = 0.2$. (a) isolines of V and the Lagrange points (red diamonds). (b) a cross-section of V at $y = 0$, showing the wells with singularities at $x = -\mu$ and $x = 1 - \mu$, as well as the Lagrange points $L1 - L3$ with $V_2 < V_1 < V_3$ (red diamonds).

heavy bodies R and the angular velocity $\omega = \sqrt{G(m_1 + m_2)/R^3}$:

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= 2v - \frac{\partial V}{\partial x} \\ \dot{v} &= -2u - \frac{\partial V}{\partial y}\end{aligned}\tag{6.12}$$

where V is the potential given by

$$V(x, y) = -\underbrace{\frac{\mu_1}{d_1}}_{(a)} - \underbrace{\frac{\mu_2}{d_2}}_{(b)} - \underbrace{\frac{1}{2}(x^2 + y^2)}_{(c)} - \underbrace{\frac{\mu_1\mu_2}{2}}_{(d)}$$

The potential comprises four different terms: (a) the potential of mass 1; (b) the potential of mass 2; (c) the centripetal force due to rotating frame of reference; and (d) a constant to make maximum value of the potential independent of μ . The landscape generated by V is interesting due to the competition between the two attracting gravitational forces (a,b) and the repelling centripetal force (c). In Figure 6.17, the potential function is shown for $\mu = m_2/(m_1 + m_2) = 0.2$ (consistent with the values displayed in table Table 1.1), and note that $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$. The singularities in the potential are at the positions of the two heavy bodies $\mathbf{x}_1 = (-\mu, 0)$ and $\mathbf{x}_2 = (1 - \mu, 0)$.

It is straightforward to establish that the system (6.12) is conservative by calculating its divergence:

$$\nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_u}{\partial u} + \frac{\partial f_v}{\partial v} = 0$$

and thus the system will not have an attractor. The total energy of the system is a constant of motion

$$H(x, y, u, v) = \frac{1}{2}(u^2 + v^2) + V(x, y)$$

which by definition remains constant along a trajectory as a function of time. This allows for a geometrical interpretation of the system; a trajectory with initial energy H_0 will remain confined to the three-dimensional subspace in the four-dimensional phase space for which $H(x, y, u, v) = H_0$. The potential V can be used to assess which region of the phase-space is accessible for a trajectory with a given $H = H_0$. Indeed, the system will by definition not be able to access the region for which $V > H_0$, which is called the *Hill region*.

The extrema on the surface of V are positioned at locations for which both $\partial V/\partial x = 0$ and $\partial V/\partial y = 0$. Lagrange discovered that for the restricted three-body problem there exist five distinct extrema, which are commonly referred to as the Lagrange points $L1 - L5$. These are indicated by diamonds in Figure 6.17. The Lagrange points $L1-L3$ are positioned on the x-axis and are numbered by convention as indicated in Figure 6.17(b). A stability calculation shows that these are all saddle points. The Lagrange point $L1$ is the most important point for our purposes: its potential V_{L1} dictates whether a satellite with energy H_0 is able to escape from the gravitational pull of the two bodies. Indeed, for $H_0 < V_{L1}$ the trajectory cannot escape. The Lagrange point $L2$ with $V_{L2} < V_{L1}$ dictates whether the satellite's orbit must remain confined to one of the two heavy bodies. There exists an energy band $V_{L2} < H_0 < V_{L1}$ for which the satellite can orbit both bodies whilst it is still unable to escape. Finally, a satellite with energy $V_{L1} < H_0 < V_{L3}$ is able to escape the system on the side of the light body only.

The equation governing the position of the Lagrange points $L1 - L3$ is a fifth order polynomial that does not have a closed-form solution, although several approximations have been developed over the years. However, it is straightforward to determine these numerically using maple, by recognising that the two heavy bodies separate the Lagrange points $L1 - L3$:



Maple

```
restart;
s1 := (x,y) -> sqrt((x+mu2)^2 + y^2);
s2 := (x,y) -> sqrt((x-mu1)^2 + y^2);
V := (x,y) -> - mu1 / s1(x,y) - mu2 / s2(x,y) - (x^2+y^2)/2 - mu1*mu2/2;
H := (x,y,u,v) -> (u^2+v^2)/2 + V(x,y);
mu1 := 1-mu; mu2 := mu;
mu := 0.2;

L1 := op(maximize(V(x,0),x=1-mu..infinity, location)[2]):
L1[1] := rhs(op(L1[1])): L1;
L2 := op(maximize(V(x,0),x=-mu..1-mu, location)[2]):
L2[1] := rhs(op(L2[1])): L2;
L3 := op(maximize(V(x,0),x=-infinity..-mu, location)[2]):
L3[1] := rhs(op(L3[1])): L3;
```

For $\mu = 0.2$, maple returns $(x_{L1}, x_{L2}, x_{L3}) = (1.27, 0.44, -1.08)$ and associated potentials $(V_{L1}, V_{L2}, V_{L3}) = (-1.86, -1.98, -1.68)$.

One of the typical features of conservative system is that there are regions of the phase space in which the trajectories are periodic or quasi-periodic that are surrounded by regions in which the system displays chaotic behavior, similar to the standard map discussed in section 3.4.6. In Figure 6.18 the trajectories for four different initial conditions are shown. In Figs Figure 6.18(a-c), $H_0 < V_{L1}$ and the trajectories orbit one of the two bodies. For the initial condition shown in Figure 6.18(d) the trajectory has an energy $V_2 < H_0 < V_1$ and as can be seen the trajectory revolves around both bodies.

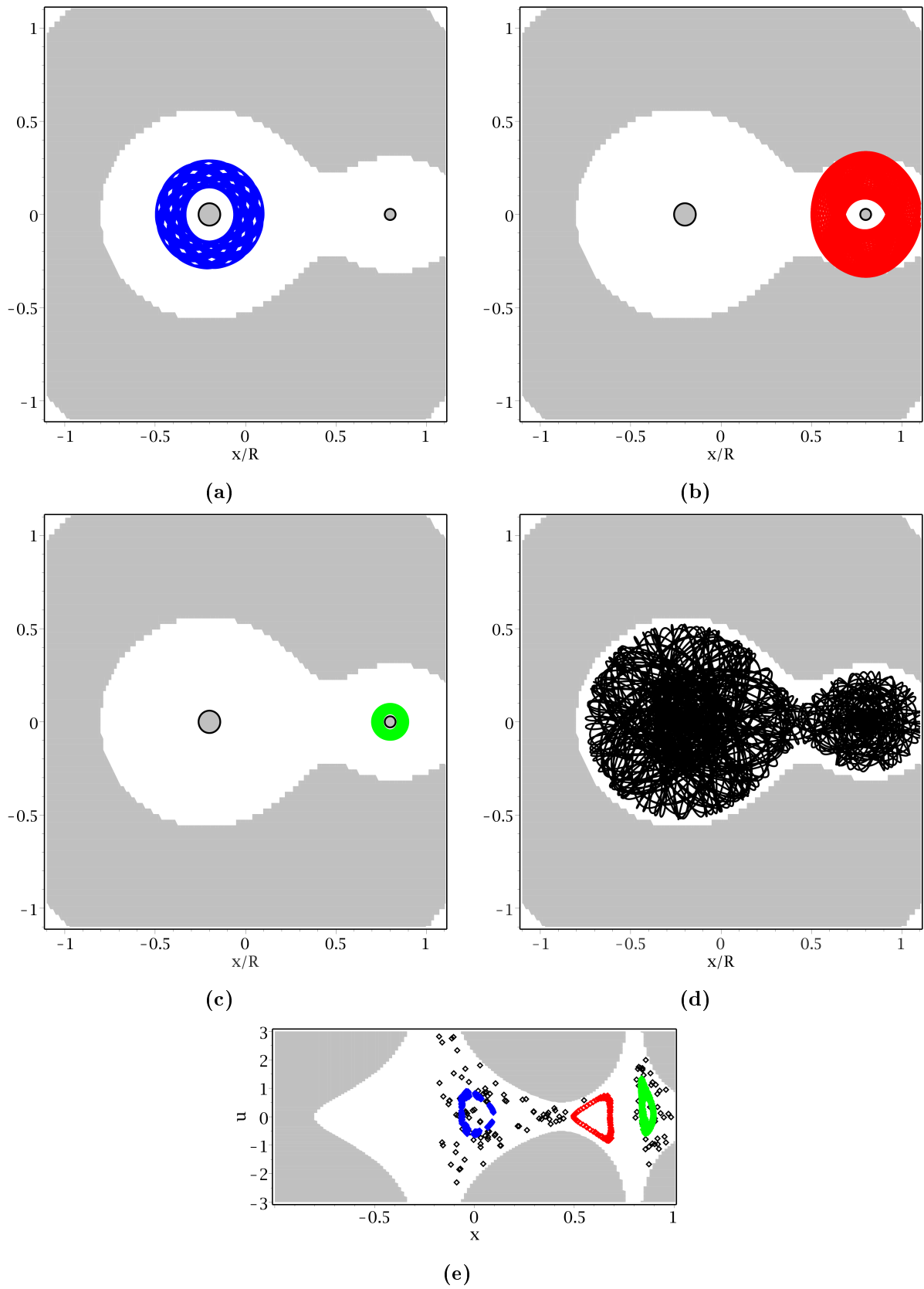


Figure 6.18: . The behaviour of the three-body system at $\mu = 0.2$ for four different initial conditions. (a-d) the trajectories in space. (e) (x_n, u_n) projection of a Poincaré section at $y = 0$ (colours correspond to different initial conditions). The Hill region $V > V_1$ is denoted in grey. (e) Poincaré section.

We construct a Poincaré section for the plane $y = 0$ and consider only those crossings for which $v > 0$, thereby ensuring that only full revolutions are considered. The Poincaré section reduces the 4-D continuous system to a 3-D discrete system, of which we show the (x_n, u_n) projection in Figure 6.18(e). Note that as conservative systems do not have an attractor, there is no need to discard transients, and all the data can be used. The orbits of Figs Figure 6.18(a-c) correspond to quasi-periodic orbits as is clear from the development of a closed loop in the Poincaré section. The behaviour of the orbit shown in Figure 6.18(d) is entirely different, as the system visits very large regions in the phase space. This is a chaotic trajectory. Another interesting point is that the chaotic regions are intertwined with the (quasi)-periodic regions as is evident from the orbits of Figs Figure 6.18(a,c).

Conservative systems represent a vast area of research and the observations above give a flavour of the dynamics. For more information we refer to e.g. [Ott, 1993, Schuster, 1995, Tél and Gruiz, 2006].

 Maple

```

xdot := (x, y, u, v) -> u:
ydot := (x, y, u, v) -> v:
udot := (x, y, u, v) -> 2*v + x - mu1*(x+mu2)/s1(x,y)^3 - mu2*(x-mu1)/s2(x,y)^
↪3:
vdot := (x, y, u, v) -> -2*u + y - mu1*y/s1(x,y)^3 - mu2*y/s2(x,y)^3:
DE1 := diff(x(t),t) = xdot(x(t),y(t),u(t),v(t)):
DE2 := diff(y(t),t) = ydot(x(t),y(t),u(t),v(t)):
DE3 := diff(u(t),t) = udot(x(t),y(t),u(t),v(t)):
DE4 := diff(v(t),t) = vdot(x(t),y(t),u(t),v(t)):
DE := DE1, DE2, DE3, DE4;
mu := 0.2;

ic := NULL;
ic := ic, [x(0) = 0.1, y(0) = 0, u(0) = 0, v(0) = 1]:
ic := ic, [x(0) = 0.5, y(0) = 0, u(0) = 0, v(0) = 1]:
ic := ic, [x(0) = 0.9, y(0) = 0, u(0) = 0, v(0) = 1]:
ic := ic, [x(0) = 0.45, y(0) = 0.1, u(0) = 0.1, v(0) = -0.1];

N := 30000; dt := 0.01;
tP := Array(1..N, i -> (i-1)*dt):
col := [blue, red, green, black]:
for nIC from 1 to nops(ic) do
    sol_arr := dsolve({DE, op(ic[nIC])}, [x(t), y(t), u(t), v(t)],
        type=numeric, maxfun=-1, output=tP):
    soldd := sol_arr[2,1];
    TT := soldd[1..N, 1];
    XX := soldd[1..N, 2]; YY := soldd[1..N, 3];
    UU := soldd[1..N, 4]; VV := soldd[1..N, 5];

    # Create Poincare section
    Px := NULL: Pu := NULL:
    for i from N/4 to N-1 do
        if(YY[i] < 0 and YY[i+1]>0) then
            Px := Px, XX[i]; Pu := Pu, UU[i];
        end if;
    end do;

    satellite[nIC] := plot([seq([XX[i], YY[i]], i=1..N)], color=col[nIC], labels=[
↪"x", "y"]):
    p[nIC] := plot([seq([Px[i], Pu[i]], i=1..nops([Px]))], style=point,
        labels=["x", "u"], color=col[nIC]);
od:
PP := inequal(H(x,y,0,0) > L1[2], x = -1.5 .. 1.5, y = -3 .. 3, 'nolines'):
earth := disk([-mu2, 0],0.06,color=gray): # earth location
moon := disk([mu1, 0],0.03,color=gray): # moon location
for n from 1 to nops(ic) do
    display([PP, satellite[n], earth, moon]);
od;

PP := inequal(H(x,0,u,0) > L1[2], x = -1.5 .. 1.5, u = -3 .. 3, 'nolines'):
display({PP, seq(p[i], i=1..nops(ic))}, axes=boxed):
display(P4, view=[-1..1, -3..3]);

```

6.6 Exercises

6.6.1 Poincaré

Downloads: `PoincareSectionRossler.mws`

-section of the Rossler system}

Consider the Rössler system

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}\tag{6.13}$$

for $a = b = 0.2, c = 5$. Check out the `Cookbook.mws` how to obtain the numerical data in `Array` format. Integrate the system and remove the transient. (Take $dt = 0.02$).

a) Calculate a Poincaré-section by selecting those instance when the trajectories traverse the $y = 0$ plane from positive to negative. To this end loop through the data-points $n = 1 \dots N$, check when the condition $y[n-1] > 0$ and $y[n] \leq 0$ is satisfied and store the *crossing*-coordinates $x_c = x[n], z_c = z[n]$ in a new array or list. Plot the point-series thus obtained in the $x - z$ plane.

b) Display the Poincaré-section and the attractor in the same plot.

c) Repeat the procedure for different values of c between 2 and 6.

6.6.2 Earth Magnetic Field

Downloads: `EarthMagneticField.mws`

Consider the following system that was inspired on a simplified version of the equations governing the Earth's magnetic field.

$$\begin{aligned}\dot{x} &= -\frac{3}{4}x + zy \\ \dot{y} &= -by + (z - 1)x \\ \dot{z} &= 1 - xy\end{aligned}$$

with $b > 0$.

a) Show that this system is dissipative.

b) Substitute

$$b = -\frac{1}{3} + \frac{(-3 + 2k^2)^2}{12k^4} \quad k > 0$$

and show that $\{x_* = 1/k, y_* = k, z_* = 3/(4k^2)\}$ and $\{x_* = -1/k, y_* = -k, z_* = 3/(4k^2)\}$ are both fixed points of the system. These turn out to be the only fixed points.

c) Calculate the stability of the fixed points at $k = 0.72$, and $k = 0.75$.

d) Based on your stability analysis, was there a bifurcation between $k = 0.72$ and $k = 0.75$. If so, what kind?

e) Take $k = 0.72$ and calculate a numerical solution for $t = 500 \dots 1000$, starting with initial conditions in the neighbourhood of one of the fixed points: $x(0) = x_* + \epsilon, y(0) = y_* + \epsilon, z(0) = z_* + \epsilon$. Try $\epsilon = 0.5$ and $\epsilon = 1.0$ and plot the attractor in phase space.

f) Discuss the results of question [eq0101:num](#) with respect to the stability analysis conducted in question [eq0101:stab](#).

6.6.3 Belousov-Zhabotinski reaction

Downloads: `BelousovZhabotinski.mws`

The Belousov-Zhabotinski (BZ) reaction is a well-known chemical oscillator. It can be made by dissolving malonic acid and ammonium nitrate in sulfuric acid. When sodium bromate is added, the solution will oscillate between yellow and clear. The behaviour of the BZ-oscillator can be modeled by the Gaspard-Nicolis equations, which are given by

$$\begin{aligned}\dot{x} &= \frac{1}{\epsilon}(z - ax^3 + bx^2 - cx), \\ \dot{y} &= y(z + sx - l), \\ \dot{z} &= z(dz - fy - x + g).\end{aligned}$$

Here x, y, z are variables and the parameters have the values

$$\begin{aligned}a &= 0.5; b = 3.0; c = 5.0; \\ f &= 0.5; g = 0.6; l = 1.3; \\ s &= 0.3; \epsilon = 0.01,\end{aligned}$$

and $d \in [0, 1]$ is an adjustable control parameter. Use the initial conditions $x(0) = 1$, $y(0) = 1$ and $z(0) = 1$.

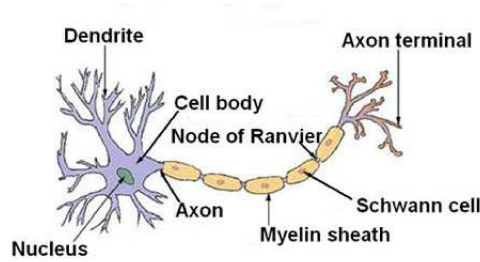
- Vary d and plot a time series of y with periodic behavior.
- Vary d and characterize the attractors by visualizing phase-space.
- Calculate the fixed-points of the system as a function of d . Complex solutions and/or solutions with `RootOf` can be dismissed.
- Use the fixed point with d -dependence and classify it for $d = 0.25$ and $d = 0.35$. What is the name of this bifurcation?
- What does it mean when a dynamical system is called dissipative? Find an appropriate measure for the 'dissipativeness' of the system and plot its time-evolution for $d = 0.5$.

6.6.4 Signal transmission in neurons

Downloads: `BVP.mws`, `ex0089r01_1.eps`

Neurons are a major class of cells in the nervous system whose main role is to process and transmit neural information. Normally, information is received through the dendrites and is sent through the axon, which can be tens to thousands of times the size of the cell body. The cell membrane in the axon and cell body contain voltage-gated ion channels which allow the neuron to generate and propagate an electrical impulse (an action potential).

The axon can be modeled by combining the equations for an excitable membrane with the differential equations for an electrical core conductor cable, which gives a set of coupled non-linear partial differential equations. When the spatial variation of the voltages is negligible, these equations reduce to a set of coupled ordinary differential equations, known as the Bonhoeffer-van der



Pol oscillator:

$$\begin{aligned}\dot{x} &= x - \frac{1}{3}x^3 - y + A + B \cos(z), \\ \dot{y} &= \frac{1}{5}x - \frac{3}{10}y, \\ \dot{z} &= \omega.\end{aligned}\tag{6.14}$$

Here x represents the voltage over the membrane and y a recovery variable. z represents the phase angle of the excitation and ω is the angular frequency. The parameter $A \in [-1, 1]$ represents the DC-bias over the membrane, and $B \in [0, \frac{1}{2}]$ the amplitude of the AC part of the excitation.

- Convert this system to its non-autonomous equivalent.
- In the following questions use the autonomous system. Take $A = 0$ and $B = 0$. Find the fixed points and determine their stability.
- Can this system with $A = 0$ and $B = 0$ feature chaotic behavior? Explain your answer.
- Take $A \in [-1, 1]$, $B = 0$. For which range of A are there three fixed points? Use `allvalues` if necessary.
- Take $A = 0$, $\omega = 1$ and $B = 0.25$. Create a plot of the solution of (6.14) in the (x, y) -phase space. As initial conditions, use $x_0 = 0, y_0 = 1, z_0 = 0$. Does this become a periodic trajectory?
- Take $A = 0$, $\omega = 1$ and $B \in [0, \frac{1}{2}]$. Create a stroboscopic Poincaré section by sampling the signal at $t_n = 2\pi n$ and plotting the (x_n, y_n) phase-space. Over which interval of B is the system chaotic?
- Calculate the effective Lyapunov exponent for $B = 0.38$.
- Discuss the relationship between the Jacobian and the effective Lyapunov exponent.

7 Fractal geometries

In the previous chapter we encountered a paradox. The Lorenz equations are dissipative, so that every volume of initial conditions in phase space shrinks to zero as $t \rightarrow \infty$. At the same time, the trajectories cannot settle onto a fixed point or a limit cycle, but are confined to the strange attractor on which they display a sensitive dependence on initial conditions. The attractor can therefore have no volume, yet it cannot be a simple surface either, as this would be incompatible with the chaotic dynamics. We resolved this paradox by claiming that the geometry of the attractor is *fractal*, with a dimension that is not an integer but lies somewhere between two and three.

The purpose of this chapter is to put this notion of a non-integer dimension on a firmer footing. We will first introduce fractals through a number of classical geometric constructions, for which the dimension can be obtained analytically from the self-similarity of the object. We then turn to the more general *box-counting dimension* and the *correlation dimension*, both of which can be determined numerically for an arbitrary set of points, such as a measured strange attractor. Finally, we consider objects for which a single dimension is not sufficient, the *multifractals*, and characterize these by a whole spectrum of *generalized dimensions* D_q .

7.1 Self-similar fractals

A geometrical object is called *self-similar* when it is built up out of smaller copies of itself. The prototypical example is the *von Koch curve*, constructed by the recursive procedure sketched in Figure 7.1. One starts with a straight line segment of unit length (the *initiator*). This segment is replaced by four segments of length $1/3$, the middle two forming a triangular bump (the *generator*). The same operation is then applied to each of the four new segments, and so on *ad infinitum*. The limiting curve is continuous everywhere but differentiable nowhere, and has an infinite length.

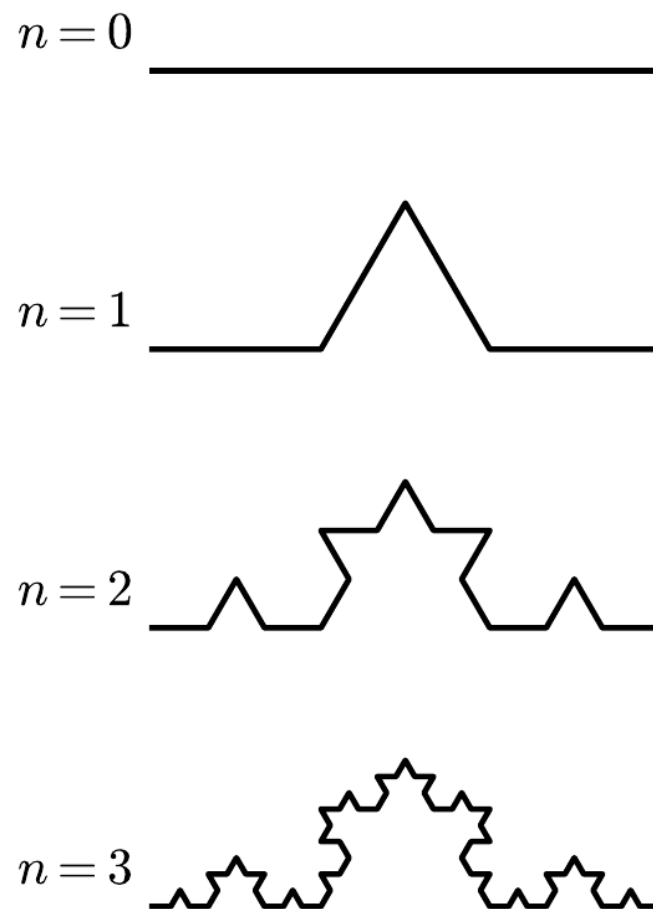


Figure 7.1: Construction of the von Koch curve. At each step every segment is replaced by four segments of one third of its length.

**Maple**

The recursion is short to code. Representing the points of the curve as complex numbers, the generator replaces each segment by four, the extra vertex being the first third rotated by $\pi/3$ to raise the bump.

```
restart: with(plots):
gen := proc(p, q) local d:                               # Koch generator for one segment
    d := (q - p)/3:
    p, p+d, p + d + d*exp(I*Pi/3), p + 2*d:             # its four leading vertices
end proc:
pts := [0, 1]:                                           # unit segment (points as complex)
for k to 4 do                                             # replace every segment, four times
    pts := [seq(gen(pts[i], pts[i+1]), i=1..nops(pts)-1), pts[-1]]:
end do:
plot([seq([Re(z), Im(z)], z in pts)], scaling=constrained);
```

To assign a dimension to such an object, we recall how the ordinary dimension of a simple geometrical shape can be characterized. Consider covering the shape with smaller copies of itself, each scaled down by a factor $s < 1$. A line segment of unit length is covered by $N = 1/s$ copies of length s ; a square by $N = (1/s)^2$ copies of side s ; a cube by $N = (1/s)^3$ copies of side s . In each case the number of copies scales as

$$N = \left(\frac{1}{s}\right)^D, \quad (7.1)$$

where D is the (integer) dimension of the object. Taking the logarithm of (7.1) and solving for D suggests the following definition of the *similarity dimension* of a self-similar object:

$$D = \frac{\log N}{\log(1/s)}, \quad (7.2)$$

where N is the number of copies and s the scaling factor of each copy. Crucially, nothing in (7.2) forces D to be an integer.

For the von Koch curve, each segment is replaced by $N = 4$ copies scaled down by $s = 1/3$, so that

$$D = \frac{\log 4}{\log 3} \approx 1.2619. \quad (7.3)$$

The dimension is larger than one, reflecting the fact that the curve is so convoluted that it starts to fill the plane, but it is smaller than two because it does not fill any area. It is precisely this kind of non-integer dimension that we anticipated for the strange attractor. By altering the generator – for instance replacing the middle bump by four equal segments set at a different angle – one can construct self-similar curves of any dimension between one and two, so that the dimension is a design parameter rather than a fixed number.

Two further classical constructions will accompany us throughout this chapter. The *Cantor set* (Figure 7.2) is obtained by repeatedly removing the middle third of a line segment: the unit interval is replaced by its two outer thirds, each of these by their outer thirds, and so on. At each stage the set consists of $N = 2$ copies scaled by $s = 1/3$, so that

$$D_{\text{Cantor}} = \frac{\log 2}{\log 3} \approx 0.6309. \quad (7.4)$$

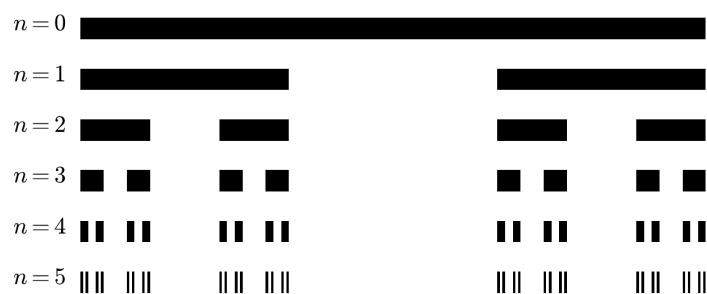


Figure 7.2: The middle-third Cantor set. At each stage the central third of every segment is removed.

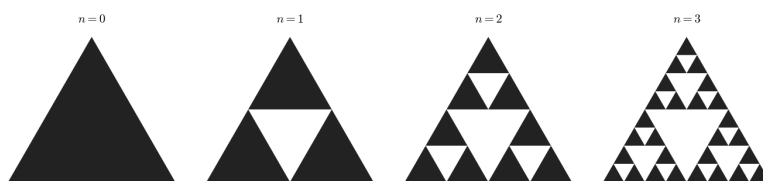


Figure 7.3: The first iterations of the Sierpinski gasket.

The Cantor set is an example of a *dust*: its dimension lies between zero (a point) and one (a line). Although it contains uncountably many points, it has zero total length.

The two-dimensional analogue is the *Sierpinski gasket* (Figure 7.3). An equilateral triangle is divided by the midpoints of its sides into four congruent sub-triangles, and the central, inverted one is removed. Each of the three remaining triangles is treated in the same manner recursively. Here $N = 3$ and $s = 1/2$, so that

$$D_{\text{Sierpinski}} = \frac{\log 3}{\log 2} \approx 1.5850. \quad (7.5)$$

As expected the dimension lies between one and two: the gasket is pierced by holes on every scale, yet its remaining filaments still branch too densely for it to be a mere curve.

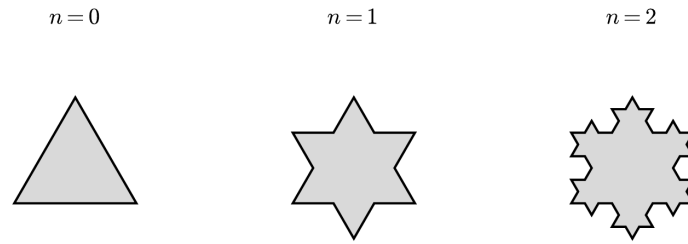


Figure 7.4: Construction of the Koch island.



Example

Self-similar reasoning can also be applied to areas and perimeters simultaneously. The *Koch island* is generated by applying the von Koch construction to each of the three sides of an equilateral triangle (Figure 7.4). The perimeter is a von Koch curve, and therefore has fractal dimension $D = \log 4 / \log 3$. If the initial perimeter has length $P_0 = 3$, then after n steps each segment has been replaced by four segments of one third the length, so that the perimeter

$$P_n = 3 \left(\frac{4}{3} \right)^n \xrightarrow{n \rightarrow \infty} \infty$$

diverges. The enclosed area, on the other hand, remains finite. At each step a new triangular bump is added to every segment; summing the resulting geometric series shows that A_n converges to $A_\infty = \frac{8}{5}A_0$. Consequently the *hydraulic diameter* $D_n = 4A_n/P_n \rightarrow 0$: the island encloses a finite area behind an infinitely long, infinitely wrinkled coastline. This is the sense in which Mandelbrot [1967] famously asked how long the coastline of Britain really is.

7.2 The box-counting dimension

The similarity dimension (7.2) is only defined for objects that are exactly self-similar. A strange attractor obtained from a numerical simulation or an experiment is not built by an explicit recursive rule, and we need a definition of dimension that can be applied to *any* set of points. The *box-counting dimension* provides exactly this, and is in this sense the more fundamental of the two: it is defined for any bounded set and is what one actually *measures* from data, while the similarity dimension is recovered as the special case in which exact self-similarity lets the boxes be counted by hand.

The idea is to cover the object with a grid of boxes of size ε and to count the number $N(\varepsilon)$ of boxes that contain at least one point of the set. For a smooth curve of length ℓ , the number of boxes needed scales as $N(\varepsilon) \sim \ell/\varepsilon = \ell \varepsilon^{-1}$; for a smooth surface of area $\mathcal{A}$ it scales as $N(\varepsilon) \sim \mathcal{A} \varepsilon^{-2}$. In general one expects

$$N(\varepsilon) \sim \varepsilon^{-D_0} \quad (\varepsilon \rightarrow 0), \quad (7.6)$$

which motivates the definition of the box-counting dimension

$$D_0 = \lim_{\varepsilon \rightarrow 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}. \quad (7.7)$$

For the exactly self-similar objects of the previous section the box-counting dimension coincides with the similarity dimension: choosing $\varepsilon = s^k$ at the k -th generation, one needs exactly $N = N_{\text{copies}}^k$ boxes, and (7.7) reproduces (7.2). The subscript 0 in D_0 anticipates the family of generalized dimensions introduced in section 7.4; the box-counting dimension is the member $q = 0$ of that family.

In practice one cannot take the limit $\varepsilon \rightarrow 0$, both because a data set contains a finite number of points and because a numerically generated fractal is only self-similar down to some smallest scale. Instead one evaluates $N(\varepsilon)$ for a range of box sizes and estimates D_0 from the slope of $\log N(\varepsilon)$ versus $\log(1/\varepsilon)$. According to (7.6), a plot of $\log N$ against $\log \varepsilon$ should be a straight line of slope $-D_0$ over the range of scales where the object is fractal.



Example

Box-counting the von Koch curve makes the yardstick idea precise. Covering the curve with boxes of size $\varepsilon = 3^{-k}$, each of the 4^k segments of the k -th generation lands in a single box, so that $N(\varepsilon) = 4^k$ and

$$D_0 = \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3} \approx 1.26,$$

exactly the similarity dimension (7.3). Equivalently, measuring the curve with a ruler of length ε returns an apparent length $L(\varepsilon) = N(\varepsilon)\varepsilon \propto \varepsilon^{1-D_0}$, which diverges as $\varepsilon \rightarrow 0$: the finer the yardstick, the longer the coastline. Running the box-counting program below on a set of points sampled along the curve recovers the same slope numerically.

The `maple` program below implements the box-counting procedure for a set of N points (x_n, y_n) in the plane, as would be obtained from the return plot of an experimental time series (here the dripping-faucet data of chapter 3). The box size is halved repeatedly by increasing the integer exponent p , so that $\varepsilon = (x_{\max} - x_{\min})/2^p$.



Maple

```

restart; with(plots): with(stats):
dat := readdata("D1.drip", [float]):
N := nops(dat)-1;
x := [seq(dat[n], n=1..N)]:
y := [seq(dat[n+1], n=1..N)]:
xmin := min(x[]): xmax := max(x[]):
ymin := min(y[]): ymax := max(y[]):
loli := NULL:
pmin := 1: pmax := 7: dp := 0.2:
for p from pmin to pmax by dp do
    delta := (xmax-xmin)/2^p;           # current box size
    ni := trunc((xmax-xmin)/delta)+1;
    nj := trunc((ymax-ymin)/delta)+1;
    nr := Matrix(ni, nj, 0);           # occupancy grid
    for n from 1 to N do               # mark every occupied box
        i := trunc((x[n]-xmin)/delta)+1;
        j := trunc((y[n]-ymin)/delta)+1;
        nr[i,j] := 1;
    end do:
    boxes := add(add(nr[i,j], j=1..nj), i=1..ni);
    loli := loli, [log(delta), log(boxes)];
end do:
xli := [seq(loli[k][1], k=1..nops([loli]))]:
yli := [seq(loli[k][2], k=1..nops([loli]))]:
eq := fit[leastsquare][[x,y], y=a*x+b]([xli, yli]); # slope = -D0

```

The final least-squares fit returns a line $y = ax + b$ with $x = \log \delta$ and $y = \log(\text{boxes})$. By (7.6) the box-counting dimension is $D_0 = -a$, i.e. minus the slope. Applying this procedure to the Sierpinski gasket of Figure 7.3 recovers the analytical value $\log 3 / \log 2$, provided that enough points are used. This last caveat is important: the box-counting method only works over the range of scales that is adequately sampled by the data. When the number of points N is too small, the finest boxes each contain at most one point, $N(\varepsilon)$ saturates at N , and the estimated slope flattens out. As a result, too few data points always lead to an *underestimate* of the dimension at small ε (Figure 7.5). A complete, self-contained version of this pipeline – generating a self-similar point set, counting occupied boxes over a range of sizes, and extracting the slope by a least-squares fit – is worked through in the *cookbook*.

The same procedure applies without modification to the strange attractors of the previous chapters, since a Poincaré section turns a continuous flow into a cloud of points in the plane. Applied to the Lorenz attractor it returns a dimension slightly above two, confirming the non-integer dimension anticipated in chapter 6 and resolving the volume-contraction paradox with which this chapter opened.

7.3 The correlation dimension

The box-counting method treats all occupied boxes on the same footing, whether a box contains a single point or a thousand. For experimental attractors it is often more robust to characterize the object through the density of pairs of points that lie close together. This leads to the *correlation dimension*.

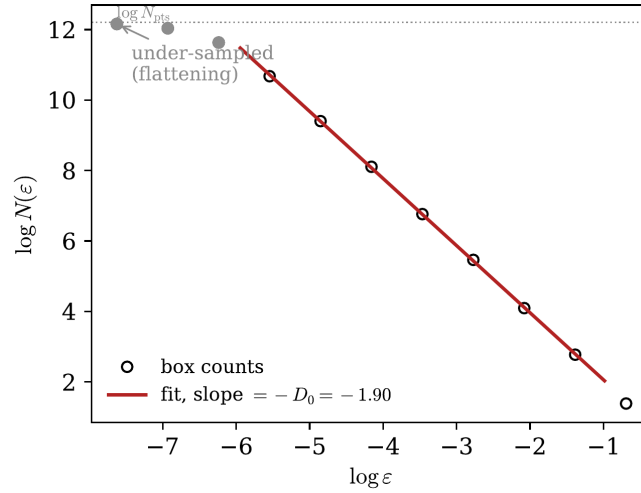


Figure 7.5: Determination of the box-counting dimension from the slope of $\log N(\varepsilon)$ against $\log \varepsilon$.

For a set of N points $\mathbf{x}_1, \dots, \mathbf{x}_N$ one defines the *correlation integral*

$$C(r) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \Theta(r - \|\mathbf{x}_i - \mathbf{x}_j\|), \quad (7.8)$$

where Θ is the Heaviside step function. In words, $C(r)$ is the fraction of all pairs of points whose separation is smaller than r . If the points are distributed on a fractal object, the number of neighbours within a distance r of a given point scales as r^{D_2} , so that

$$C(r) \sim r^{D_2} \quad (r \rightarrow 0), \quad (7.9)$$

which defines the correlation dimension

$$D_2 = \lim_{r \rightarrow 0} \frac{\log C(r)}{\log r}. \quad (7.10)$$

As with box-counting, D_2 is estimated in practice from the slope of a $\log C(r)$ versus $\log r$ plot over the scaling range. The correlation dimension is generally somewhat smaller than the box-counting dimension, $D_2 \leq D_0$, with equality only when the points are uniformly distributed over the object; it is the member $q = 2$ of the family of generalized dimensions.

The following `maple` fragment computes the correlation integral for a set of N points (x_n, y_n) . Rather than sorting distances, the pairwise separations are binned logarithmically between $r_{\min}$ and $r_{\max}$, and the bins are accumulated to form the cumulative count $C(r)$.



Maple

```

restart; with(plots):
# ... x, y and N are assumed to hold the data set ...
rmin := 0.04: rmax := 5.0: M := 20:
mumin := log(rmin): mumax := log(rmax):
dmu := (mumax - mumin)/M:
for m from 0 to M do
  cor[m] := 0;
  scale[m] := exp(mumin + dmu*(m + 0.5)); # representative r of bin m
end do:
for i from 2 to N do # loop over all pairs i>j
  for j from 1 to i-1 do
    distance := sqrt((x[i]-x[j])^2 + (y[i]-y[j])^2);
    m := trunc((log(distance + 1e-10) - mumin)/dmu);
    if (m >= 0 and m <= M) then cor[m] := cor[m] + 1; end if;
  end do:
end do:
for m from 1 to M do
  cor[m] := cor[m] + cor[m-1]; # cumulative -> C(r)
end do:
loli := seq([log(scale[m]), log(cor[m])], m=1..M):

```

The slope of a straight-line fit to `loli` yields the correlation dimension D_2 directly. Applied to the dripping-faucet attractors, this procedure gives values close to, but slightly below, the box-counting dimensions of section 7.2, consistent with the general ordering $D_2 \leq D_0$.

7.4 Multifractals and generalized dimensions

So far we have characterized a fractal object by a single number, its dimension. This is adequate when the points are distributed more or less uniformly over the object. Many attractors, however, are highly non-uniform: the trajectory spends far more time in some regions than in others, so that the *measure* is concentrated in a way that a purely geometric dimension cannot capture. Such objects are called *multifractals*, and their proper description requires not one dimension but a whole spectrum.

To make this quantitative, we cover the object with boxes of size ε as before, but now we keep track of the fraction $p_i(\varepsilon)$ of the total measure that falls in box i (for a time series, p_i is the fraction of time the trajectory spends in box i). We then form the partition sum

$$Z(q, \varepsilon) = \sum_i p_i(\varepsilon)^q, \quad (7.11)$$

where the sum runs over occupied boxes and q is a real parameter. Positive values of q emphasize the densely populated boxes, whereas negative values of q emphasize the sparsely populated ones. In analogy with (7.6) one expects a power-law scaling $Z(q, \varepsilon) \sim \varepsilon^{(q-1)D_q}$ and defines the *generalized dimension* of order q as

$$D_q = \frac{1}{q-1} \lim_{\varepsilon \rightarrow 0} \frac{\log \sum_i p_i^q}{\log \varepsilon}. \quad (7.12)$$

The prefactor $1/(q-1)$ is chosen so that D_q reproduces the dimensions we already know for the special values of q :

- For $q = 0$, $\sum_i p_i^0$ simply counts the number of occupied boxes $N(\varepsilon)$, so that D_0 is the box-counting dimension (7.7).
- For $q = 2$, $\sum_i p_i^2$ is the probability that two randomly chosen points fall in the same box, and D_2 is the correlation dimension (7.10).
- For $q \rightarrow 1$ the definition becomes indeterminate; taking the limit carefully (l'Hôpital) yields the *information dimension*

$$D_1 = \lim_{\varepsilon \rightarrow 0} \frac{\sum_i p_i \log p_i}{\log \varepsilon}, \quad (7.13)$$

in which the numerator is (minus) the Shannon entropy of the box distribution.

The generalized dimensions form a non-increasing sequence, $D_q \geq D_{q'}$ for $q < q'$. For a *monofractal*, where the measure is spread uniformly, all D_q are equal and the spectrum collapses to a single value. A genuine *multifractal* has a D_q that decreases with q : the width of the spectrum $D_{-\infty} - D_{+\infty}$ measures how non-uniform the object is.

7.4.1 A binomial multifractal

The mechanism is best understood through an explicit construction. Consider the unit interval carrying a unit of “mass”. At the first step we split the interval into two equal halves and distribute the mass in the proportion p_1 to the left half and $p_2 = 1 - p_1$ to the right half. At the second step each half is again split in two and its mass redistributed in the same proportion $p_1 : p_2$, and so on. After k steps the interval has been divided into 2^k sub-intervals of length $\varepsilon = 2^{-k}$, and the measure contained in a sub-interval reached through a particular sequence of left/right choices is a product of the form

$$p_i = p_1^m p_2^{k-m},$$

where m is the number of times the “left” branch was taken. This recursive assignment,

$$[1] \rightarrow [p_1, p_2] \rightarrow [p_1^2, p_2 p_1, p_1 p_2, p_2^2] \rightarrow \dots,$$

generates a self-similar *measure* on the interval (Figure 7.6). For $p_1 = p_2 = \frac{1}{2}$ the mass is spread uniformly and the object is monofractal, but for $p_1 \neq p_2$ it becomes increasingly spiky as k grows, with the bulk of the measure concentrated on a vanishingly small fraction of the interval.

The generalized dimensions of this measure can be obtained analytically. Because the 2^k sub-intervals realize every combination of left/right choices, the partition sum (7.11) factorizes,

$$Z(q, \varepsilon) = \sum_i p_i^q = \sum_{m=0}^k \binom{k}{m} \left(p_1^m p_2^{k-m} \right)^q = (p_1^q + p_2^q)^k,$$

where the last step uses the binomial theorem. With $\varepsilon = 2^{-k}$ we have $\log \varepsilon = -k \log 2$, and substitution into (7.12) gives the closed-form result

$$D_q = - \frac{\log (p_1^q + p_2^q)}{(q-1) \log 2}. \quad (7.14)$$

One verifies immediately that $D_0 = \log 2 / \log 2 = 1$: the measure lives on the whole interval, so the *support* is one-dimensional. The information about the non-uniformity is carried entirely by the dependence of D_q on q . For $p_1 = p_2 = \frac{1}{2}$ equation (7.14) gives $D_q = 1$ for all q (monofractal), whereas for $p_1 \neq p_2$ the dimension D_q decreases monotonically from $D_{-\infty}$ to $D_{+\infty}$, producing the characteristic sigmoidal D_q curve of Figure 7.7.

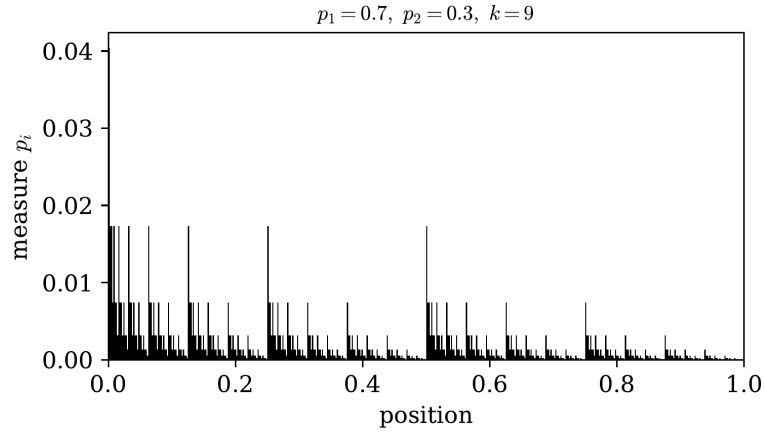


Figure 7.6: A binomial multifractal measure obtained by repeatedly redistributing the mass of each sub-interval in the ratio $p_1 : p_2$.

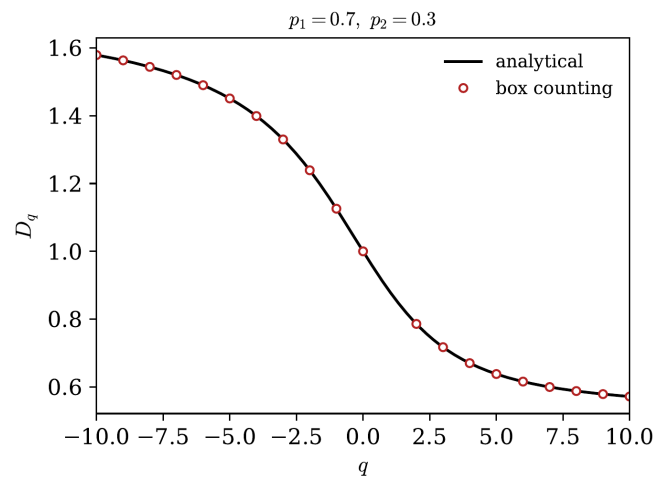


Figure 7.7: The generalized-dimension spectrum D_q of the binomial multifractal, comparing the analytical result with the box-counting procedure.

7.4.2 Numerical determination of D_q

The generalized dimensions of an arbitrary measure are computed by a straightforward extension of the box-counting algorithm. For each box size $\varepsilon = 2^{-p}$ one accumulates the measure p_i in every box, forms the partition sum $Z(q, \varepsilon) = \sum_i p_i^q$, and records $\log Z$ against $\log \varepsilon$. Repeating this over a range of box sizes and fitting a straight line gives the slope, from which $D_q = \text{slope}/(q - 1)$ follows for each q .



Maple

```
restart: with(stats):
read "p1p2.dat":           # loads the measure dat[1..N]
N := nops(dat);
x := [seq(n/N, n=1..N)]:
listDq := NULL:
for q from -10 to 10 do
  if q = 1 then next; end if;   # q=1 is singular, skip it
  loli := NULL:
  for p from 2 to 9 do
    delta := 1.0/2^p;
    nb := 2^p;
    nr := Array(1..nb, 0.0);
    for n from 1 to N do       # bin the measure
      i := trunc(x[n]/delta) + 1;
      nr[i] := nr[i] + dat[n];
    end do:
    Z := add(if(nr[i] > 0, nr[i]^q, 0), i=1..nb);
    loli := loli, [log(delta), log(Z)];
  end do:
  xli := [seq(loli[k][1], k=1..nops([loli]))]:
  yli := [seq(loli[k][2], k=1..nops([loli]))]:
  eq := fit[leastsquare][[x,y], y=a*x+b]([xli, yli]);
  Dq := coeff(rhs(eq), x)/(q-1);   # slope / (q-1)
  listDq := listDq, [q, Dq];
end do:
plot([listDq], labels=["q", "D(q)"]);
```

The value $q = 1$ is skipped in the loop because of the $1/(q - 1)$ singularity; the information dimension (7.13) would have to be evaluated separately. Comparing the numerical spectrum with the analytical expression (7.14) (Figure 7.7) provides a stringent test of the box-counting implementation, and shows the same under-sampling limitations at large $|q|$ that we encountered for D_0 : extreme values of q weight either the most or the least populated boxes, both of which are the most poorly sampled.

**Example**

The construction generalizes to any number of branches. Splitting each interval into three parts with weights p_1, p_2, p_3 leads, by the same argument, to

$$D_q = - \frac{\log(p_1^q + p_2^q + p_3^q)}{(q-1) \log 3}. \quad (7.15)$$

Setting one of the weights to zero prunes the corresponding branch at every stage, and the support of the measure becomes a Cantor-like dust rather than the full interval. For instance, with $p_2 = 0$ and $p_1 = p_3 = \frac{1}{2}$ the measure is uniform on the two surviving thirds, and (7.15) reduces for $q = 0$ to $D_0 = \log 2 / \log 3$, exactly the dimension (7.4) of the middle-third Cantor set. Choosing instead $p_2 = 0$, $p_1 = \frac{4}{5}$, $p_3 = \frac{1}{5}$ gives a genuine multifractal supported on the Cantor set, whose spectrum D_q again follows (7.15).



Fractal geometry of turbulence

Fractals are not merely mathematical constructions; turbulent flows produce them in abundance. A striking example is the *turbulent/non-turbulent interface* (TNTI), the sharp but intricately folded surface that separates the rotational, turbulent fluid in a jet, wake or gravity current from the irrotational fluid around it [Corrsin and Kistler, 1954]. It is convoluted across the whole range of eddy sizes, from the integral scale l_o down to a few Kolmogorov lengths l_i , and box-counting (section 7.2) of the interface area yields a power law [Mandelbrot, 1982, Sreenivasan *et al.*, 1989]

$$A_\eta(\lambda) \propto \lambda^{2-D_f}, \quad l_i \lesssim \lambda \lesssim l_o,$$

where D_f is the fractal dimension of the surface. A planar cut through it is a fractal curve whose length scales as $L(\lambda) \propto \lambda^{-\beta}$ with $\beta = D_f - 2$; high-Reynolds-number flows give $\beta \approx 1/3$, i.e. $D_f \approx 7/3$ [de Silva *et al.*, 2013, Sreenivasan, 1991]. This matters because the interface area controls *entrainment*: the more convoluted the interface, the faster the turbulence engulfs and mixes the surrounding fluid.

What sets β ? A self-similar model in the spirit of section 7.1 is revealing (Figure 7.8). A segment of length l_n is replaced by two segments meeting at a vertex displaced a distance $r l_n$ off its midpoint, the side alternating so that the mean position is preserved. Each new segment has length $l_{n+1} = \frac{1}{2} m l_n$ with $m = \sqrt{1 + 4r^2}$, and there are twice as many, so the curve consists of $N = 2$ copies of itself scaled by $s = m/2$. Its dimension then follows straight from the similarity dimension (7.2),

$$D = \frac{\log 2}{\log(2/m)}, \quad \beta = D - 1 = \frac{\log m}{\log(2/m)}.$$

This is the von Koch construction with a tunable bump: $r \approx 0.32$ reproduces $\beta = 1/3$, whereas smaller r gives a flatter, lower-dimensional interface.

The anisotropy parameter r – the height of the interface bulges relative to their width – is where the physics enters. In a stratified gravity current a stable density gradient suppresses the vertical excursions of the interface while its streamwise extent stays fixed by the shear layer, so r decreases as the stratification (measured by the Richardson number) grows. Through the relation above this lowers β , shrinks the interface area A_η , and throttles the entrainment of ambient fluid [Krug *et al.*, 2017]; in the limit $\beta \rightarrow 0$ the interface becomes smooth and mixing collapses to molecular diffusion. The fractal dimension is thus a genuine control parameter for turbulent mixing in the ocean and atmosphere. (Should r itself vary with scale, the interface is no longer a simple fractal but a multifractal of the kind met in section 7.4.)

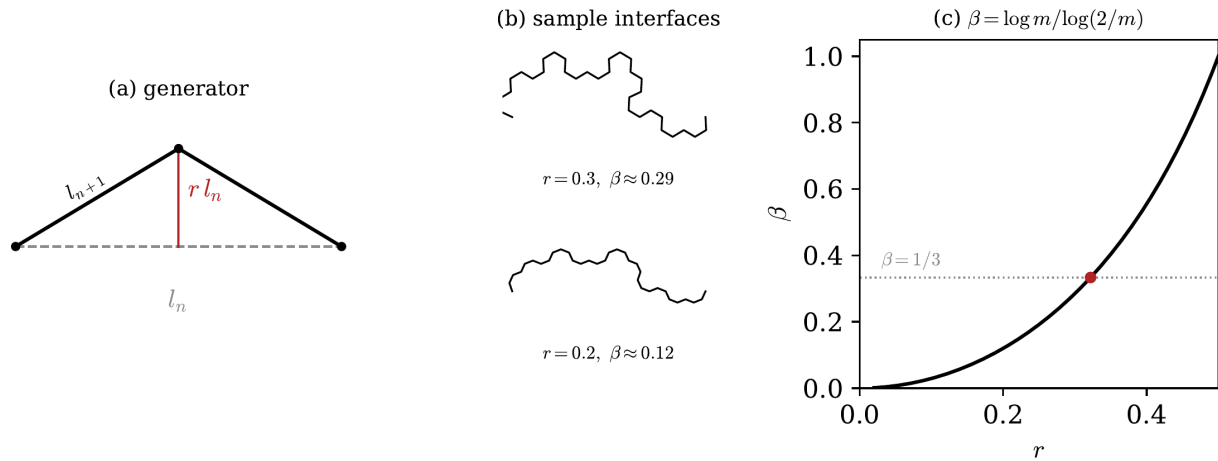


Figure 7.8: A self-similar model of the turbulent/non-turbulent interface [Krug2017]. (a) The generator: a segment of length l_n is replaced by two segments meeting at a vertex displaced by $r l_n$ off the midpoint. (b) Curves after five iterations for $r = 0.3$ and $r = 0.2$; the smaller r produces a smoother, lower-dimensional interface. (c) The fractal exponent $\beta = \log m / \log(2/m)$ as a function of the anisotropy r , with the neutral-turbulence value $\beta = 1/3$ marked.



Fractal clouds

The boundary of a cumulus cloud is one of the most familiar fractals in Nature. Satellite images show that the cloud perimeter has a fractal dimension close to $4/3$ over four orders of magnitude in size which, assuming isotropy, means the cloud *surface* scales as $A \sim L^{D_s}$ with $D_s \simeq 7/3$. Siebesma and Jonker [2000] tested this with a computer model of the atmosphere (a *large-eddy simulation*) and measured the dimension of individual simulated clouds using exactly the tools of this chapter: the volume–surface scaling $A \sim L^{D_s}$ and the correlation integral (7.8). Both gave $D_s = 2.3\dots$, in striking agreement with the observations (Figure 7.9).

The value $7/3$ is no accident. It is precisely the surface dimension for which the turbulent transport of heat and moisture across the (unresolved) cloud edge becomes *independent* of the grid resolution – so it is the fractal geometry of the interface that makes the mixing, or *entrainment*, come out right. The fractal cloud boundary is really the turbulent/non-turbulent interface of the previous box, now seen from the sky.

7.5 Exercises

7.5.1 Correlation dimension

Downloads: CorrelationDimension.mws

- Write a maple program to calculate the *correlation dimension* of a set consisting of N points (x_n, y_n) .
- Calculate the correlation dimension of the dripping faucet attractors and compare the values with the box-dimension.

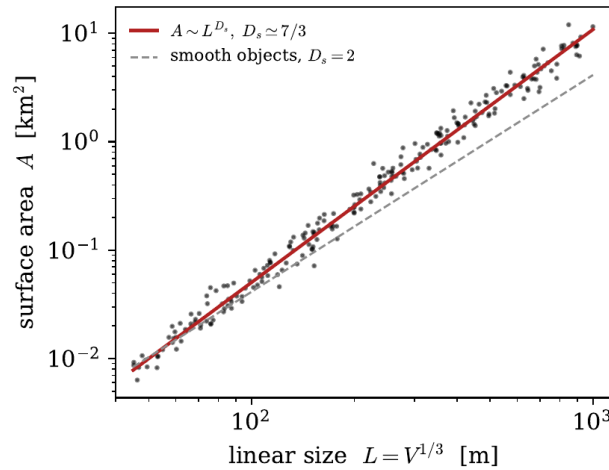


Figure 7.9: Surface area A against linear size $L = V^{1/3}$ for a population of simulated cumulus clouds (schematic, after Siebesma & Jonker, 2000). The clouds obey $A \sim L^{D_s}$ with $D_s \simeq 7/3$ (solid line), well above the $D_s = 2$ of smooth, Euclidean shapes (dashed line).

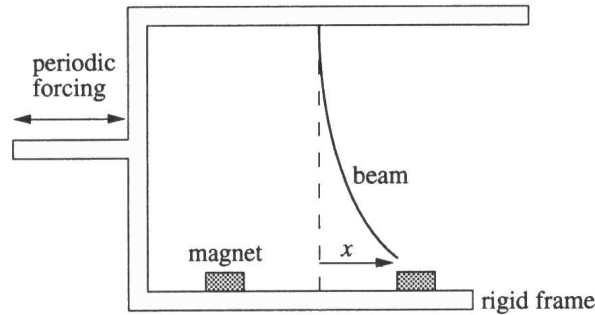


Figure 7.10: Sketch of the mechanical system.

7.5.2 Fractal Properties of the Forced Double Well Oscillator

Downloads: `DoubleWellFractal.mws`, `DoubleWellFractal_start.mws`, `ex0121r00_fig1.eps`, `ex0121r00_fig2.eps`

Consider a slender steel beam with a damping coefficient f that is clamped in a rigid framework (Figure 7.10). Two permanent magnets at the base pull the beam in opposite directions and the magnets are so strong that the beam buckles to one side or the other. The system is driven out of its equilibrium by shaking it with angular frequency ω and force A .

Written as an autonomous system of first order nonlinear differential equations, the system's behavior is defined by

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -fy + x - x^3 + A \cos z, \\ \dot{z} &= \omega.\end{aligned}\tag{7.16}$$

a) Open the file `DoubleWellFractal_start.mws`, and execute it for $A = 0.3$. This generates output like shown in Figure 7.11. What does this code-fragment do? What is the name of this plotting method? Why and when would this be useful?

b) Determine the box-dimension of this attractor.

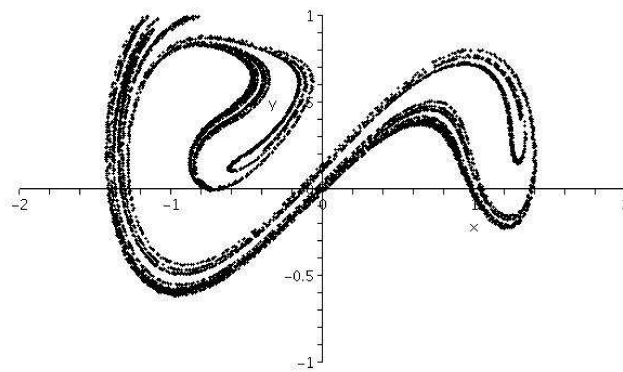


Figure 7.11: Result from executing `DoubleWellFractal\_start.mws` with $N = 10000$.

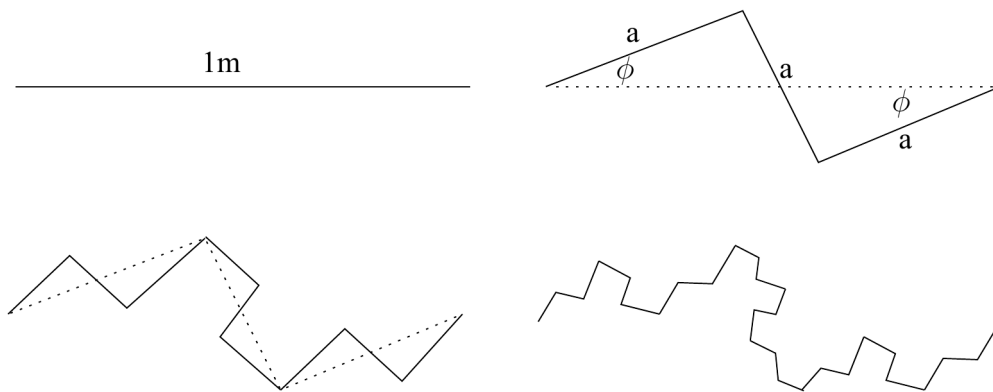


Figure 7.12: Fractal Curve generation: first four recursive steps

7.5.3 Fractal Curve

Downloads: `FractalCurve.mws`, `FractalCurve.data`, `FractalCurve_start.mws`, `FractalCurve_creator.mws`, `FractalCurve.eps`

Consider the above sketched recursive generation of a family of fractal curves parameterized by ϕ , with $0 \leq \phi \leq \pi/6$. Note that the three segments have equal size $a = a(\phi)$ and that $a \cos(\phi) \leq 1/2$.

- Express the fractal dimension d of the curve in terms of ϕ and plot the result.
- Take $\phi = \pi/7$. After how many recursive steps is the length of the curve larger than 1km?
- The file `FractalCurve.data` contains the points of a certain curve generated along the way outline above. Open the maple file `FractalCurve_start.mws` which reads this data set, and calculate the curve's fractal dimension using the box-counting method.

7.5.4 Multi-fractals in threes

Downloads: `MultifractalThree.mws`

- Generate a multi-fractal set with $N = 3^6$ points, according to the formalism: $[1] \rightarrow [p_1, p_2, p_3] \rightarrow [p_1^2, p_1 p_2, p_1 p_3, p_1 p_2, p_2^2, p_2 p_3, p_1 p_3, p_2 p_3, p_3^2] \rightarrow \dots$
- Take $p_2 = 0$, and $p_3 = p_1 = 1/2$. Calculate analytically the fractal dimension (the box dimension).

- c) Take $p_2 = 0$, $p_1 = 4/5$ and $p_3 = 1/5$. Calculate analytically the generalized dimension $D(q)$.
- d) Take $p_2 = 0$, $p_1 = 4/5$ and $p_3 = 1/5$. Calculate the generalized dimension $D(q)$ by the box-counting procedure for q in the range $q \in [-10, 10]$ (avoid calculating $q = 1$ in the procedure). Plot the result and compare the outcome with the analytical result obtained in (c). (Hint: divide the boxes each time in three parts.)

8 Chaos in practice

The book began with a paradox – that simple, entirely deterministic equations can produce behaviour that is impossible to predict in the long run – and has since assembled the geometric machinery needed to understand it: phase space and flows, fixed points and their stability, bifurcations and the routes to chaos, Lyapunov exponents, strange attractors and, in the previous chapter, their fractal geometry. All of this was developed on deliberately small systems, from one-dimensional maps to the three equations of Lorenz, precisely so that every idea could be computed and drawn.

The purpose of this final chapter is to look outward. The same handful of concepts – a finite horizon of predictability, a cloud of trajectories in place of a single one, an attractor of far lower dimension than the space it lives in – are not merely of historical interest. They are the working machinery of modern computational science, from weather forecasting and climate reanalysis to the reduced-order modelling of turbulence and the analysis of measured signals. We shall introduce no new mathematics. Instead we show that the reader has acquired a *way of thinking* about nonlinear systems, and follow it along two complementary lines. Because prediction has a finite horizon, we must *predict with ensembles, correct with data, and reduce with structure*; and where chaos is an asset rather than an obstacle, we can turn it to advantage – to mix, to control, and to reconstruct. Throughout, the Lorenz equations of chapter 6 serve as a concrete laboratory in which each idea can still be run on a desktop.

8.1 Prediction in an uncertain world

It is worth pausing to see why a weather forecast belongs in this book at all. A numerical weather model solves the equations of motion of the atmosphere – conservation of mass, momentum and energy for a rotating, stratified, moist fluid – on a grid covering the globe. Collecting every field value (three velocity components, temperature, pressure, humidity, . . .) at every grid point into a single enormous column vector $\mathbf{x}$, the model is nothing other than a system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{8.1}$$

of exactly the form (6.1) that we have studied since chapter 5. The only difference is one of scale: where the Lorenz system has three components, a modern forecast model has a state vector $\mathbf{x}$ whose dimension is of order 10^9 (Figure 8.1).

This is more than an analogy. As shown in appendix B, the Lorenz equations were themselves obtained by Galerkin-truncating a convection model until only three modes remained. The tiny system the reader now knows intimately and the billion-dimensional weather model are therefore *the same construction* seen at two extremes of dimension. Everything the book has established

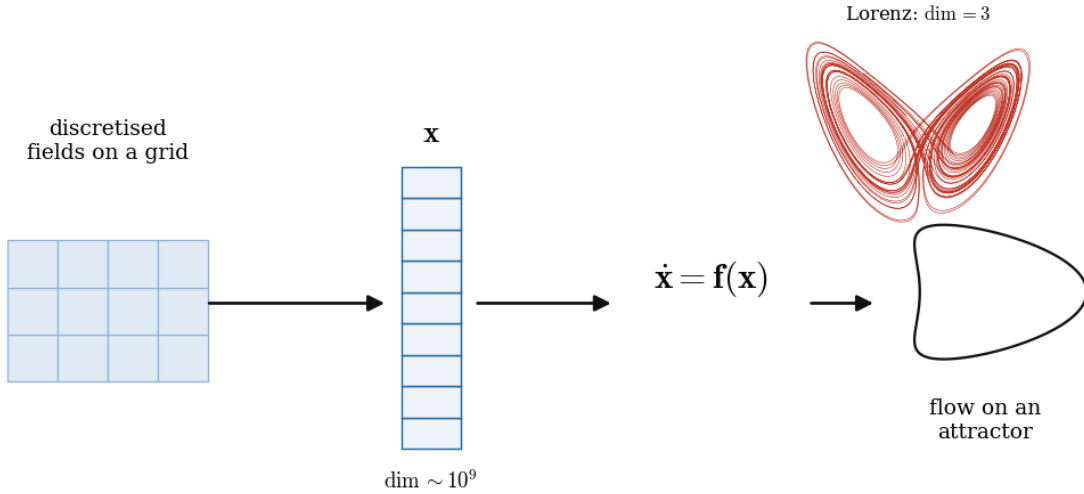


Figure 8.1: A weather model is a dynamical system. The discretised atmospheric fields are stacked into one state vector $\mathbf{x}$ of dimension of order 10^9 , whose evolution obeys $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. The Lorenz system is the same kind of object at the opposite extreme of dimension.

about (8.1) carries over: the forced–dissipative atmosphere is drawn onto an attractor – the *climate* – on which the ever-changing *weather* is nothing but the trajectory itself; the motion on that attractor has a positive Lyapunov exponent, and hence a sensitive dependence on initial conditions. It was in contemplating exactly this that Lorenz was led to his three equations, and it is this that makes weather prediction hard. The rest of the chapter simply pushes the familiar ideas up in dimension.

The first of these ideas concerns the limits of prediction. The central lesson of chaos is not that prediction is impossible, but that every prediction has a *finite horizon*. Two trajectories that start a distance δ_0 apart separate, on average, as $\delta(t) \sim \delta_0 e^{\Lambda_e t}$, where Λ_e is the effective Lyapunov exponent of chapter 6. A forecast remains useful only until this error grows to the size of the attractor itself. For the Lorenz system, with $\Lambda_e \approx 0.9$ (section 6.1), an initial uncertainty of $\delta_0 = 10^{-3}$ grows to order ten in a time

$$t_h \approx \frac{1}{\Lambda_e} \log \frac{\delta_{\text{att}}}{\delta_0} \approx \frac{1}{0.9} \log \frac{10}{10^{-3}} \approx 10,$$

after which the forecast retains no information about the specific initial condition. The atmosphere behaves in exactly the same way; it is only the numbers that differ, giving the familiar horizon of a week or two for weather. Its effective growth rate is not a single constant, however, but varies with the flow regime and with the spatial scale of the feature one is trying to predict, so this horizon is itself only a typical value.

Since the initial state is never known exactly, the honest response is to abandon the single deterministic forecast and to propagate instead a whole *cloud* of initial conditions, each an equally plausible estimate of the true state. This is the idea of *ensemble forecasting*. Rather than one trajectory, one integrates many, and reads off the forecast not as a single number but as the distribution of the ensemble. The butterfly effect thereby ceases to be a curiosity and becomes an operational quantity: the *spread* of the ensemble is the forecast's own estimate of its uncertainty.

Figure 8.2 shows a Lorenz ensemble: a tight blob of initial conditions is stretched into a filament along the attractor and, by $t = 8$, smeared across it entirely. The blob's growth is quantified in

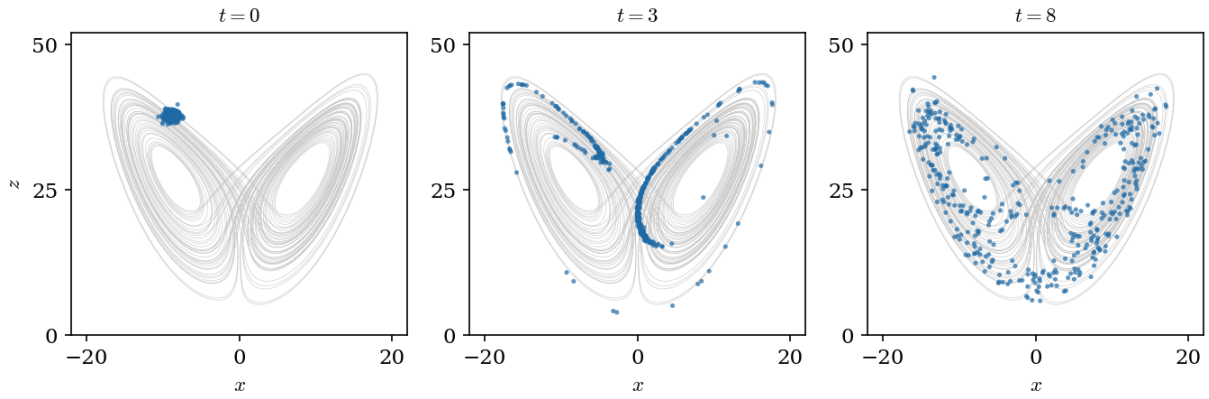


Figure 8.2: An ensemble forecast for the Lorenz system, shown in the (x, z) projection. A tight blob of initial conditions (left) is stretched into a filament by the flow, and eventually spread over the whole attractor. The width of the cloud is the forecast uncertainty.

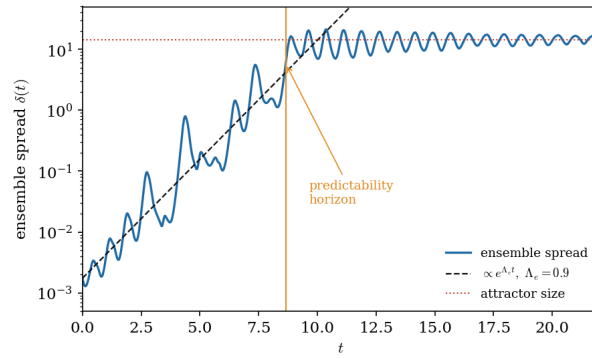


Figure 8.3: Growth of the ensemble spread $\delta(t)$ for the blob of Figure 8.2. The spread grows as $e^{\Lambda_e t}$ with $\Lambda_e = 0.9$ (dashed) until it saturates at the size of the attractor (dotted). The predictability horizon is reached when the two meet.

Figure 8.3, which tracks the ensemble spread $\delta(t)$. It grows exponentially at the rate $\Lambda_e = 0.9$ set by the Lyapunov analysis, until it saturates at the size of the attractor. The time at which the spread becomes an appreciable fraction of the attractor is the predictability horizon: beyond it, the ensemble members are effectively uncorrelated and only the climatological distribution remains.

Maple

The ensemble is generated by integrating the Lorenz equations from a set of perturbed initial conditions and measuring how the cloud spreads. Starting from a point $\mathbf{x}_0$ already on the attractor (obtained as in chapter 6 by discarding a transient), we draw M nearby initial conditions and integrate each.

```
restart; with(Statistics): with(LinearAlgebra):
sigma := 10: r := 28: b := 8/3:
DE := diff(x(t),t) = sigma*(y(t)-x(t)),
      diff(y(t),t) = r*x(t) - y(t) - x(t)*z(t),
      diff(z(t),t) = x(t)*y(t) - b*z(t):

M      := 400:                               # ensemble size
eps0   := 1e-3:                               # initial spread
for m from 1 to M do                          # perturb the initial state
  ic := x(0)=x0[1]+eps0*Sample(Normal(0,1),1)[1],
        y(0)=x0[2]+eps0*Sample(Normal(0,1),1)[1],
        z(0)=x0[3]+eps0*Sample(Normal(0,1),1)[1]:
  sol[m] := dsolve({DE, ic}, numeric, output=listprocedure):
end do:

for i from 1 to N do                          # spread at each time t[i]
  cloud := Matrix(M, 3):
  for m from 1 to M do
    cloud[m,1] := eval(x(t), sol[m])(t[i]):
    cloud[m,2] := eval(y(t), sol[m])(t[i]):
    cloud[m,3] := eval(z(t), sol[m])(t[i]):
  end do:
  cbar := <Mean(cloud[..,1]), Mean(cloud[..,2]), Mean(cloud[..,3])>:
  delta[i] := sqrt(add(Norm(<cloud[m,1..3]> - cbar, 2)^2, m=1..M)/M):
end do:
```

Ensemble weather forecasting

Operational weather centres run precisely this calculation, but with $\dim \mathbf{x} \sim 10^9$ instead of three. Because one cannot integrate millions of members, a modest ensemble of some 50 forecasts is launched from slightly perturbed initial conditions and slightly perturbed model physics [Bauer *et al.*, 2015, Lorenz, 1996]. The spread of the ensemble at a given lead time *is* the forecast uncertainty: when the members agree, the forecast is confident; when they diverge, it is not. This is how a probability of rain is produced, and it is why forecasts now come with a horizon attached. The transformation of weather prediction from a single deterministic run into a routine estimate of its own uncertainty has been called a quiet revolution [Bauer *et al.*, 2015]; at its root lies nothing more than the sensitive dependence on initial conditions of chapter 6.

8.2 Observations and models: data assimilation

A forecast needs a starting point, and here a second difficulty appears: the true state $\mathbf{x}$ is never known. We have instead a stream of noisy, incomplete *observations* – a scatter of measurements that constrains the state without determining it. Forecasting in practice is therefore a continuous cycle: *predict* the state forward with the model, *observe* it where measurements are available, and *correct* the prediction towards the observations, before predicting again. Combining a model with

data in this way is called *data assimilation*, and it is once more an application of the dynamical system (8.1).

The classical tool for the correction step is the *Kalman filter* [Kalman, 1960]. Its central idea is disarmingly simple: the corrected estimate, called the *analysis*, is a weighted average of the model forecast and the observation,

$$\mathbf{x}_a = \mathbf{x}_f + K(\mathbf{y} - H\mathbf{x}_f), \quad (8.2)$$

where $\mathbf{x}_f$ is the forecast, $\mathbf{y}$ the observation, and H the operator that maps a state to the quantities actually measured. The quantity $\mathbf{y} - H\mathbf{x}_f$ is the *innovation*, the amount by which the observation surprises the forecast, and the *Kalman gain* K is the weight applied to it. The gain is chosen by the relative confidence in model and data: writing P for the covariance (the uncertainty) of the forecast and R for that of the observations,

$$K = PH^\top (HPH^\top + R)^{-1}. \quad (8.3)$$

When the observations are precise (R small) the gain is large and the analysis follows the data; when the model is trusted (P small) the gain is small and the analysis stays close to the forecast. The filter alternates two steps: a *forecast* step, in which both the state and its uncertainty P are advanced by the model, and an *update* step (8.2)–(8.3), in which the incoming observation shrinks that uncertainty.

Advancing the uncertainty is the only subtlety. The textbook Kalman filter is linear, whereas $\mathbf{f}$ is not, so the covariance is propagated through the *linearisation* of the flow – and that linearisation is precisely the Jacobian J already used to compute the Lyapunov exponents in chapter 6. This is the *extended* Kalman filter. No new machinery is needed: over one assimilation window the forecast covariance evolves as $P_f = \Phi P_a \Phi^\top + Q$, where the state-transition matrix Φ is obtained by integrating $\dot{\Phi} = J\Phi$ with $\Phi(0) = I$, and Q accounts for imperfections in the model.

Figure 8.4 shows the result of a *twin experiment* on the Lorenz system: we integrate a "truth" trajectory, generate noisy observations from it, and then try to recover it from a deliberately wrong initial guess. A free forecast, left to run without corrections, diverges from the truth within the predictability horizon of the previous section. The Kalman filter, fed the same noisy observations, locks onto the truth and tracks it indefinitely, its error held an order of magnitude below the free run. The finite horizon has not been abolished – it cannot be – but by repeatedly injecting information it is continually reset before the error can grow to saturation.

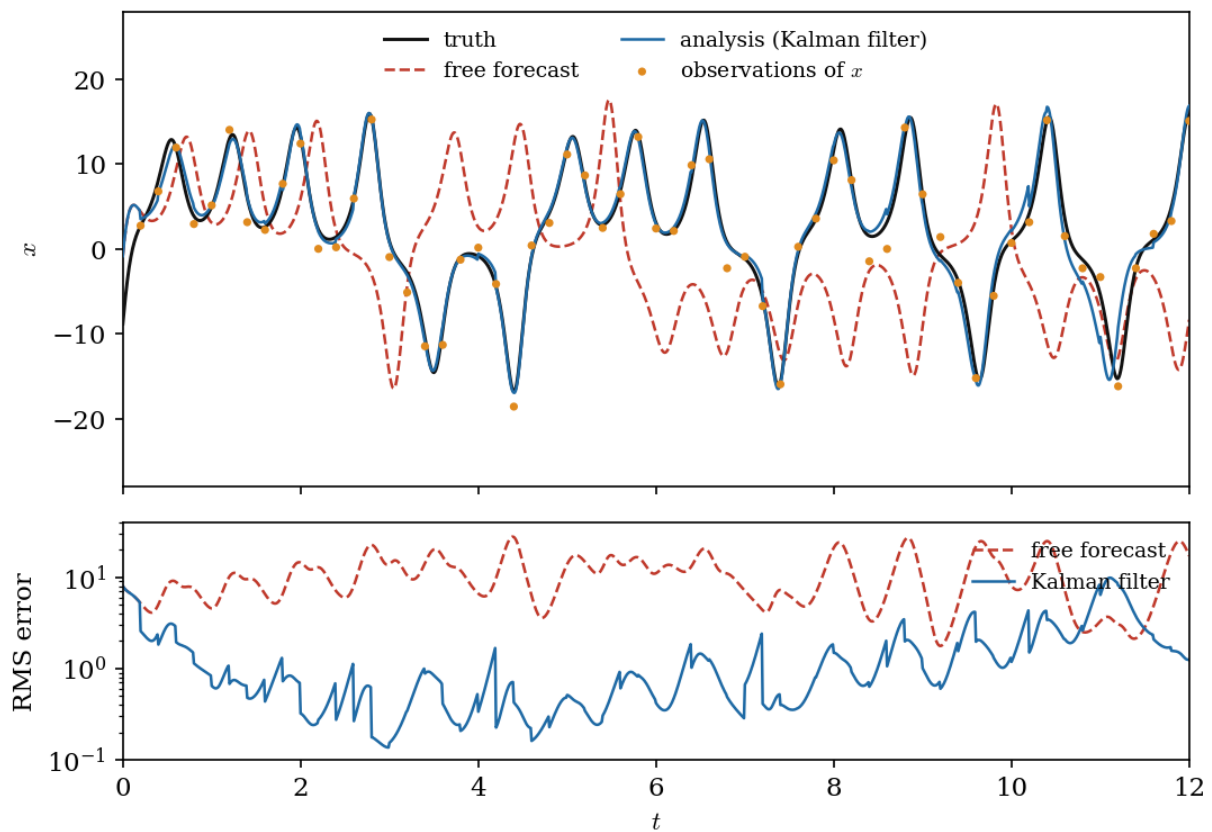


Figure 8.4: Data assimilation for the Lorenz system. Top: the x -component of the truth, a free forecast started from a wrong initial condition, and the Kalman-filter analysis fed noisy observations (dots). Bottom: the RMS error of the two estimates. The free forecast diverges within the predictability horizon; the analysis stays locked on.



Maple

The filter is a short loop over the observation times. Between observations the state is advanced with `dsolve` and the covariance with the Jacobian `Jac`; at each observation the analysis and its covariance are updated using the Kalman gain (8.3). The Jacobian is the same one used for the Lyapunov exponents in chapter 6.

```
restart; with(LinearAlgebra):
sigma := 10: r := 28: b := 8/3:
f := (x,y,z) -> <sigma*(y-x), r*x - y - x*z, x*y - b*z>:
Jac := (x,y,z) -> <<-sigma, r-z, y> |
                  < sigma, -1, x> |
                  < 0, -x, -b>>: # as in chapter cont3d

H := IdentityMatrix(3): # full state observed
R := (1.5)^2 * IdentityMatrix(3): # observation error
Q := 0.01 * IdentityMatrix(3): # model error
I3 := IdentityMatrix(3):

xa := <-5.0, -5.0, 25.0>: # wrong first guess
Pa := 4.0 * IdentityMatrix(3): # its uncertainty

for k from 1 to Nobs do
  # --- forecast: advance state and covariance over one window ---
  xf := advance(xa, dtobs): # integrate the ODE
  Phi := transition(xa, dtobs): # solve d(Phi)/dt = Jac.Phi
  Pf := 1.03 * Phi . Pa . Transpose(Phi) + Q: # (mild inflation)
  # --- update: blend the forecast with observation y[k] ---
  S := H . Pf . Transpose(H) + R:
  K := Pf . Transpose(H) . MatrixInverse(S): # Kalman gain
  xa := xf + K . (y[k] - H . xf): # analysis
  Pa := (I3 - K . H) . Pf: # reduced uncertainty
  Xa[k] := xa:
end do:
```

Because the Lorenz system has only three variables, its error covariance P is a 3×3 matrix that can be stored and propagated exactly. This is what makes the example transparent, but it is also exactly what is *impossible* in an operational model, where P would be a $10^9 \times 10^9$ matrix that can be neither stored nor propagated. The extended Kalman filter is thus the natural nonlinear extension in principle but is unusable for global forecasting in practice, and overcoming this is the central technical problem of modern data assimilation. Two families of practical methods have emerged. *Ensemble* methods replace the explicit covariance with the sample covariance of a modest ensemble (section 8.1), giving the *ensemble Kalman filter* [Evensen, 2003], supplemented by localisation and inflation to control sampling error. *Variational* methods, on which the major operational centres principally rely, instead fit the model trajectory to a whole window of observations by numerical optimisation (the technique known as 4D-Var), and are increasingly combined with ensembles into hybrid schemes [Bauer *et al.*, 2015]. The Kalman filter of (8.2) is best regarded as the conceptual prototype for all of these rather than the operational algorithm itself; what they share is unchanged from it – predict, observe, and correct in proportion to confidence.

Run this same cycle continuously on a model of one *specific* engineered system – an aircraft engine, a bridge, a patient's circulation – kept synchronised to its physical counterpart by a live stream of sensor data, and one has what engineering now calls a *digital twin*: data assimilation applied not to the planet but to a single machine, and often accelerated by the reduced-order surrogates

of the next section.



Reanalysis: replaying the atmosphere

Apply the same assimilation cycle to the historical record and one obtains a *reanalysis*. The model is run *forward* through decades of past observations, its state corrected within each assimilation window to be consistent with the measurements collected there – surface stations, radiosondes, aircraft, satellites. The ERA5 reanalysis [Hersbach and others, 2020] does this for the whole atmosphere since 1940 using incremental 4D-Var, combining model forecasts with the millions of observations of the global observing network into a statistically optimal, physically consistent reconstruction of the past atmosphere comprising hundreds of millions of prognostic variables. It rests on the same Bayesian philosophy as (8.2) – combining an imperfect model with imperfect data – scaled up to the whole planet, and it underlies much of what is quantitatively known about the changing climate.

8.3 Finding simplicity in complexity

The billion dimensions of a weather model are, in a sense, an illusion. The system is strongly dissipative, so its trajectories collapse onto an attractor of far lower dimension than the space that contains them – exactly as the Lorenz flow, living in three dimensions, is confined to an attractor of fractal dimension only slightly above two (chapter 7). More generally, the essential dynamics of a high-dimensional dissipative system often unfold on a low-dimensional *manifold* (Figure 8.5), and the practical art is to find coordinates that expose it. This is the aim of *reduced-order modelling*. The picture should be read with some care: the precise object may be an attractor, an inertial, centre or slow manifold, and for a system as intricate as turbulence one can rarely prove that the dynamics collapse onto a single smooth low-dimensional manifold. What is robust – and what reduced-order modelling exploits – is that the *effective* number of active degrees of freedom is far smaller than the dimension of the state vector.

The oldest and most widely used of these methods is *proper orthogonal decomposition* (POD), known in other fields as principal component analysis. Given a collection of snapshots of a field – successive velocity fields of a turbulent flow, say – POD finds the fixed spatial patterns, or *modes*, ordered so that the first captures as much of the variance (the energy) as possible, the second as much of the remainder, and so on [Berkooz *et al.*, 1993]. For flows with coherent structure, the energy is concentrated in a mere handful of modes, and the field can be reconstructed from them to good accuracy (Figure 8.6). The high-dimensional field is thereby reduced to the evolution of a few modal amplitudes – a small dynamical system of the kind studied in this book.

POD organises a field by energy, but says nothing about its dynamics. A complementary idea is to seek modes that evolve simply *in time*. *Dynamic mode decomposition* (DMD) extracts from a sequence of snapshots the patterns that grow, decay and oscillate at definite rates [Schmid, 2010], and is closely tied to the *Koopman* viewpoint (Figure 8.7): although the flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is nonlinear in the state, it acts *linearly* on functions of the state, the *observables* [Mezić, 2005]. In exchange for moving to a higher-dimensional space of observables, one recovers a linear description – with all the eigenvalues and modes that linearity brings – of a nonlinear system.

POD, DMD and the Koopman operator are best seen not as competing techniques but as three expressions of a single idea: *finding low-dimensional structure in a high-dimensional system*. It is this idea, rather than any particular algorithm, that endures – and it is a direct descendant of the observation, made throughout this book, that dissipative chaotic systems live on attractors far smaller than their phase space.

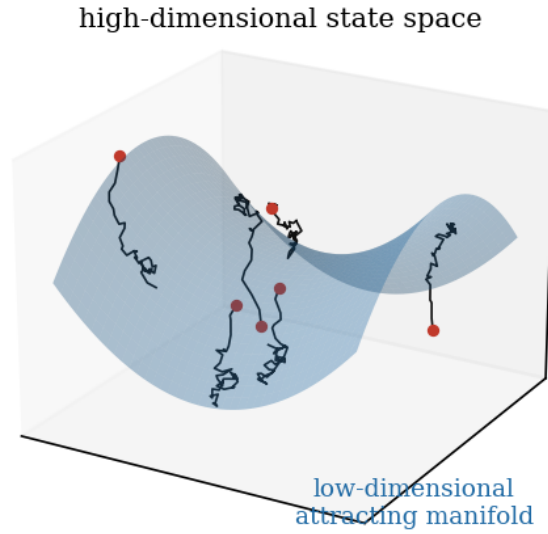


Figure 8.5: Trajectories of a dissipative system, launched from scattered initial conditions in a high-dimensional state space, collapse onto a low-dimensional attracting manifold. Reduced-order models seek coordinates on this manifold.

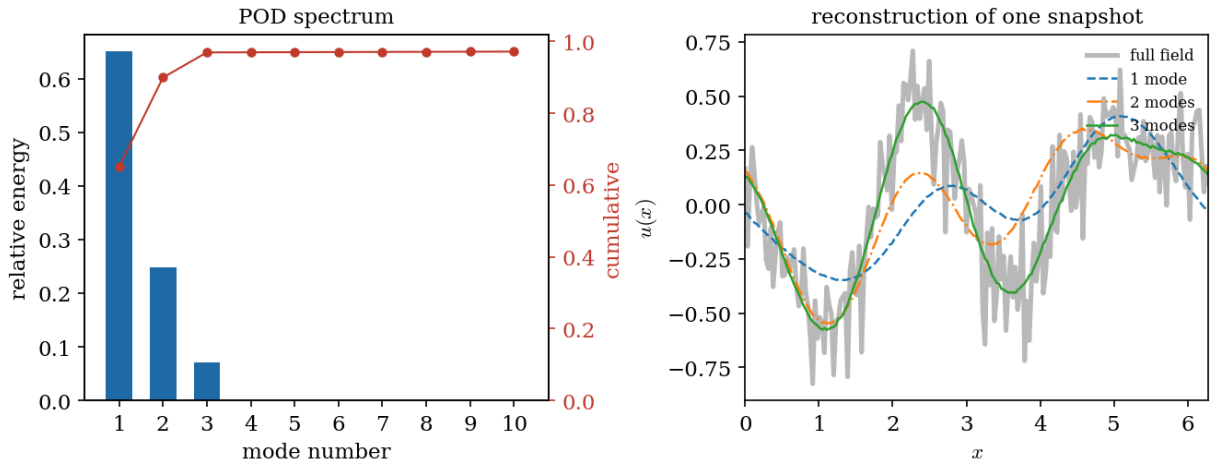


Figure 8.6: Proper orthogonal decomposition of a field with low-dimensional structure. Left: the energy is captured almost entirely by the first three modes. Right: a single snapshot, reconstructed with one, two and three modes, converges rapidly to the full field; the residual is noise.

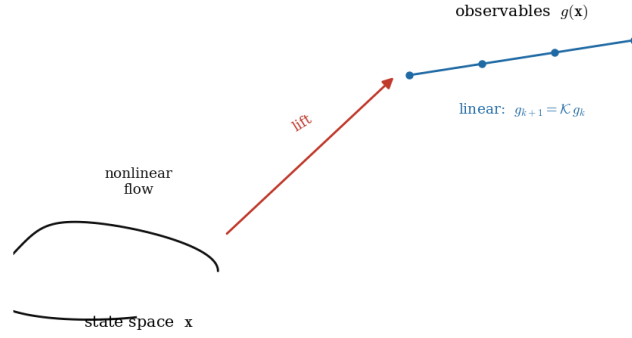


Figure 8.7: The Koopman viewpoint. The flow is nonlinear in the state $\mathbf{x}$, but by lifting to a space of observables $g(\mathbf{x})$ it can be represented by a linear operator $\mathcal{K}$. Dynamic mode decomposition approximates this operator from data.

The same idea underlies the more recent, and more fashionable, use of machine learning in dynamical systems. It is best regarded not as a replacement for the theory of this book but as one more way to do two familiar things: to *approximate* the nonlinear operator $\mathbf{f}$ of (8.1) when it is unknown or too expensive to evaluate, and to *discover* low-dimensional coordinates for a complex system when the linear modes of POD are not flexible enough. A neural-network surrogate that emulates an expensive model is doing the former; an autoencoder that compresses a high-dimensional field to a few latent variables is doing the latter, and its latent space is simply a nonlinear version of the attracting manifold of Figure 8.5. Presented this way, the methods are variations on the themes of the previous sections rather than a new subject, and we do not dwell on them here; the monograph of Brunton and Kutz [2019] develops the connection between machine learning, dynamical systems and control at length.

8.4 Harnessing chaos

The chapter so far has treated chaos as an adversary – something to be predicted around, corrected for, or compressed away. Yet the very properties that make a chaotic system hard to predict can also be put to use. Sensitive dependence stretches and folds, which is exactly what is needed to *mix*; the dense set of unstable orbits buried in an attractor makes it exquisitely *controllable*; and the way a single variable carries the imprint of the whole attractor lets us *reconstruct* the dynamics from one measured signal. We take these three in turn.

Stirring milk into coffee mixes it in seconds, but at the small scales of a microfluidic channel the flow is smooth and slow, and molecular diffusion alone would take an age. The way out is that a fluid element's *position* can behave chaotically even when the flow itself is simple and laminar – this is *chaotic advection* [Aref, 1984, Aref and Balachandar, 1986]. A convenient model advects each element (x_n, y_n) on the unit square through one period of a time-periodic flow by two successive shears,

$$\begin{aligned} x_{n+1} &= x_n + A \sin(2\pi y_n), \\ y_{n+1} &= y_n + A \sin(2\pi x_{n+1}), \end{aligned} \tag{8.4}$$

with both coordinates taken modulo one so that the fluid stays on the square. This is a close relative of the standard map of chapter 3: each shear merely slides fluid along one direction and so preserves area, as an incompressible flow must, yet for an amplitude A of order one the map stretches and folds a blob of dye into ever finer filaments (Figure 8.8), until diffusion across the

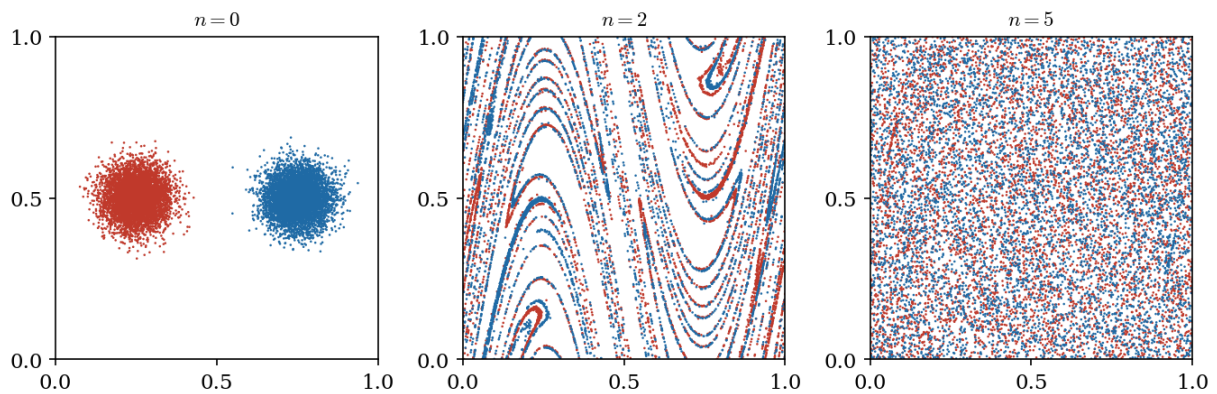


Figure 8.8: Chaotic mixing. Two blobs of dye advected by a simple time-periodic flow are stretched and folded into ever finer filaments and, within a few periods, interleaved throughout the domain – efficient mixing produced by a smooth, laminar flow.

thin lamellae finishes the job. Shaping a real channel so that its flow is chaotic in this way is now a standard route to efficient micromixers for drug delivery and diagnostics, and the same mechanism stirs the mantle, the oceans and the atmosphere.



Maple

The map (8.4) is just the two shears applied in turn; `wrap` keeps the dye on the unit square by taking each coordinate modulo one. Iterating it on a thin streak of points reproduces the filaments of Figure 8.8.

```
restart: with(plots):
A := 0.9: wrap := t -> t - floor(t):           # stay on the unit torus
step := proc(p) local xn:                     # one period of the flow
  xn := wrap(p[1] + A*sin(2*Pi*p[2])):        # horizontal shear
  [xn, wrap(p[2] + A*sin(2*Pi*xn))]:         # vertical shear
end proc:
blob := [seq([0.25, 0.40 + 0.2*k/100], k=0..100)]: # a thin streak of dye
for n to 5 do blob := map(step, blob): end do:
pointplot(blob, symbol=point, view=[0..1, 0..1]);
```



Coherent structures and ocean transport

The same template scales up to the planet. In the ocean and atmosphere the stretching and folding organise themselves around hidden curves and surfaces – the *Lagrangian coherent structures* – that act as the transport barriers and conduits of the flow, the moving skeleton along which tracers are funnelled [Haller, 2015]. Computed from satellite-derived surface currents, they predict where an oil slick will spread or where a person lost at sea will drift; attracting structures were found to coincide with the observed extent of the *Deepwater Horizon* spill, and the same calculation now guides search-and-rescue operations and the tracking of drifting plastic. The fractal, folded interfaces of chapter 7 and these coherent structures are two views of the one stirred, chaotic flow.

The second use turns sensitivity into control. A chaotic attractor is far from structureless: threaded through it is a dense set of *unstable periodic orbits*, each a perfectly regular motion that the system shadows briefly before sensitive dependence throws it off. Because the trajectory passes close to any chosen orbit sooner or later, and because sensitive dependence means a *tiny*

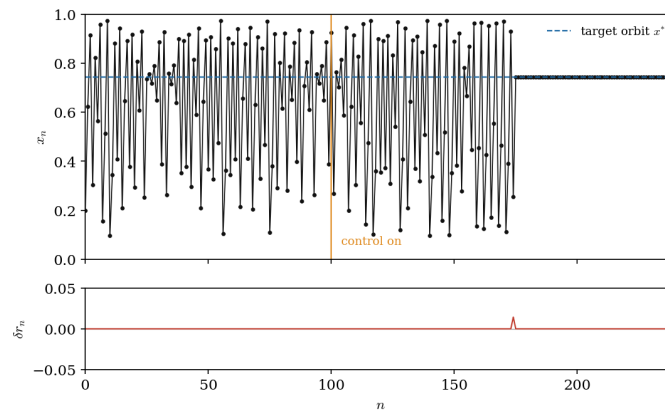


Figure 8.9: Controlling chaos. Left of the marker the logistic map ($r = 3.9$) iterates chaotically. Once control is enabled it waits until the orbit strays near the target fixed point x^* and then, with the minuscule parameter nudges δr_n shown below, locks the motion onto it.

nudge has a large effect, one can stabilise that orbit with vanishingly small, well-timed adjustments of a parameter – the method of Ott *et al.* [1990]. Figure 8.9 demonstrates it on the logistic map $x_{n+1} = r x_n(1 - x_n)$ of chapter 2. The target is its unstable fixed point $x^* = 1 - 1/r$, and the control at step n is the change in growth rate that would send the current state exactly there; solving $(r + \delta r_n) x_n(1 - x_n) = x^*$ gives

$$\delta r_n = \frac{x^*}{x_n(1 - x_n)} - r, \quad (8.5)$$

applied only while it stays small, $|\delta r_n| \leq \delta r_{\max}$. Because the orbit repeatedly strays near x^* , and a per-cent-level nudge is then enough to catch it, the once-chaotic iteration is captured onto the fixed point and held there indefinitely. A regular system would need a large control to be moved so far; it is precisely the chaos that makes such delicate steering possible. The same idea has been used to suppress the arrhythmic beating of cardiac tissue [Garfinkel *et al.*, 1992] and to tame the flickering output of lasers.



Maple

Controlling the logistic map costs only a few lines: at each step, if the state is near the target x^* , nudge r by the small amount (8.5) that would map it exactly onto the fixed point.

```
restart: with(plots):
r := 3.9: xstar := 1 - 1/r: drmax := 0.05:  # target orbit and largest nudge
x := 0.2: orbit := NULL:
for n to 240 do
  rr := r:
  if n > 100 then                                # switch the controller on
    dr := xstar/(x*(1-x)) - r:                  # nudge that would hit x*
    if abs(dr) <= drmax then rr := r + dr: end if:
  end if:
  x := rr*x*(1-x):
  orbit := orbit, [n, x]:
end do:
plot([orbit], style=point, symbolsize=6);
```

The third use reads chaos from data. In an experiment one seldom measures the full state; more often a single scalar is recorded – a voltage, a velocity at one point, the interval between drips of

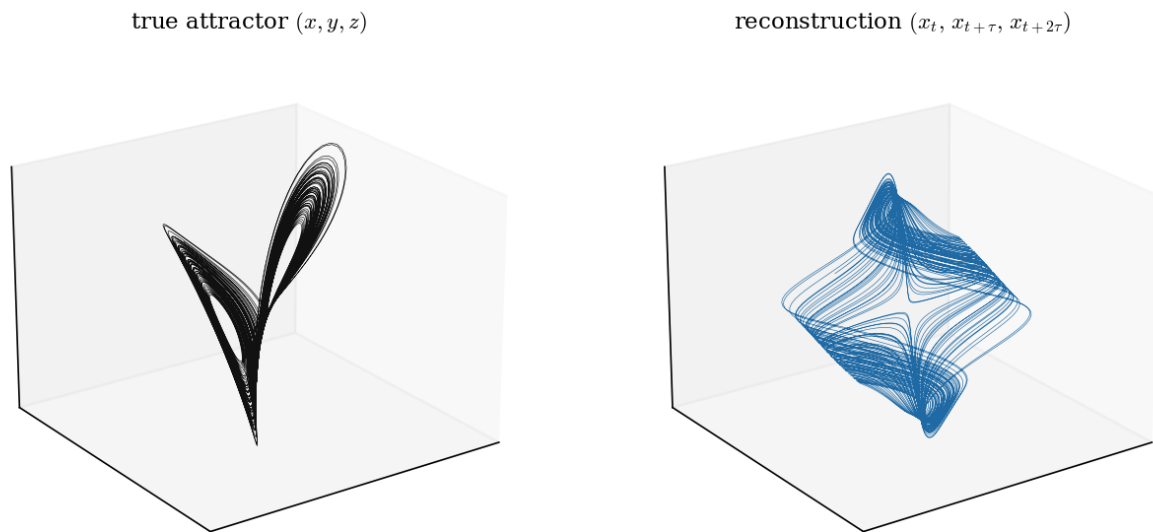


Figure 8.10: Attractor reconstruction. From the single recorded coordinate $x(t)$ of the Lorenz system (left), the delay-coordinate vectors $(x_t, x_{t+\tau}, x_{t+2\tau})$ trace out a reconstruction (right) with the same two-lobed topology as the original – the content of Takens' theorem.

the faucet of chapter 3. Remarkably, that lone time series is enough. *Takens' theorem* [Takens, 1981] guarantees that the vectors built from delayed copies of one coordinate, $(x_t, x_{t+\tau}, x_{t+2\tau}, \dots)$, trace out a faithful copy of the original attractor, with the same dimension, Lyapunov exponents and topology. Figure 8.10 rebuilds the Lorenz attractor from its x -coordinate alone. This *delay embedding* is what lets the box-counting and correlation dimensions of chapter 7 be measured from a real signal, and it underpins the analysis of chaotic data across science – detecting the approach of an epileptic seizure from an EEG, diagnosing incipient faults in rotating machinery, and forecasting ecological and physiological records.



Maple

Reconstruction needs only a single recorded coordinate. If the list X holds the sampled values $x(t_i)$ of a Lorenz run, the delay vectors are formed and plotted directly.

```
restart: with(plots):
tau := 18:                                     # delay, in samples
N := nops(X) - 2*tau:
emb := [seq([X[i], X[i+tau], X[i+2*tau]], i=1..N)]:
pointplot3d(emb, symbol=point, symbolsize=1, axes=boxed);
```

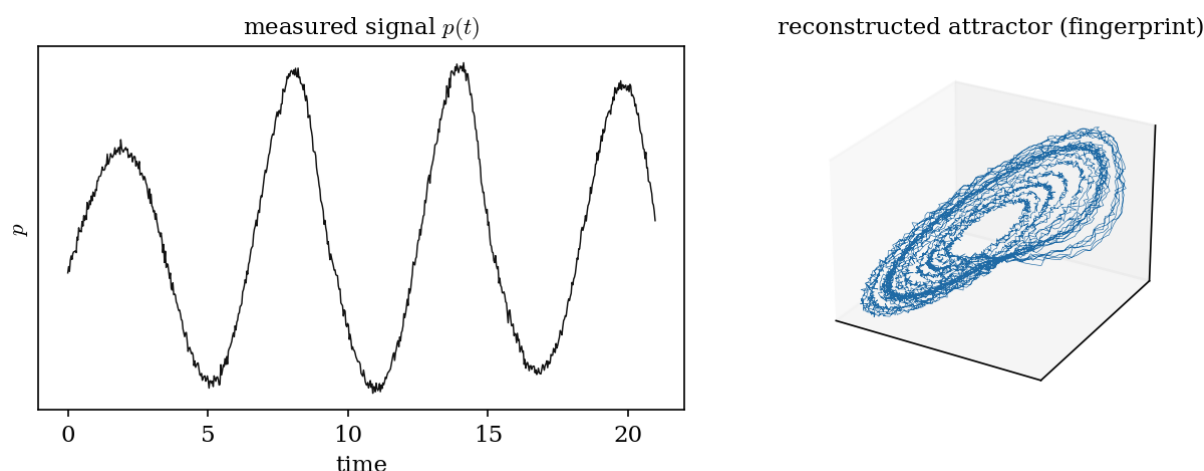


Figure 8.11: The Takens route in industry (illustrative). A single measured signal – here a noisy chaotic time series standing in for the pressure in a fluidised-bed reactor (left) – is turned by delay embedding into a reconstructed attractor (right). The shape of this "fingerprint" characterises the operating regime, and changes in it give early warning of malfunction.



Fingerprinting a chemical reactor

The delay-embedding idea is not confined to the laboratory. Industrial *fluidised beds* and *bubble columns* – workhorses of the chemical industry – are strongly nonlinear and, in many operating regimes, chaotic. A single, cheaply measured pressure signal is enough to reconstruct their attractor, and the shape of that attractor, together with a measure of its unpredictability (the Kolmogorov entropy), serves as a *fingerprint* of the hydrodynamic state (Figure 8.11) [van den Bleek *et al.*, 2002]. The chaotic-processes group at Delft – the very setting from which this book grew, with Floris Takens himself involved – turned this fingerprint into a suite of industrial tools: characterising flow regimes and their transitions, guiding the scale-up of reactors by matching attractors, and even *controlling* the bubbling. A small, energy-neutral feedback adjustment of the gas supply forces the bed onto a chosen unstable periodic orbit – the control idea of this section applied to a reactor – and was shown to raise the yield of a catalytic reaction by more than ten per cent. The same fingerprint drives an early-warning system: by continuously comparing the live attractor with a healthy reference one, the onset of particle *agglomeration* is caught long before a conventional pressure-drop alarm would trip, in time to keep the reactor from clogging solid.

8.5 Why nonlinear dynamics matters

The applications sketched here are far from solved problems. Predicting the coupled ocean–atmosphere system across the many scales of climate, closing the equations of turbulence, modelling living systems and steering large engineered networks all remain among the hardest challenges in computational science. What they share is the difficulty this book has been about from the start: they are nonlinear, high-dimensional and sensitive to their initial conditions, so that brute-force prediction fails and one must instead reason about ensembles, attractors and reduced descriptions.

We began by asking how a system as simple as a one-dimensional map or three coupled equations could defeat prediction. In answering it, you have acquired something more general than

a catalogue of chaotic phenomena. The finite horizon that forces us to forecast with ensembles, the correction of models by data, the low-dimensional structure hidden inside high-dimensional systems, and the recognition that the same sensitivity can be turned to advantage – to mix, to control, to reconstruct – are not facts about the Lorenz equations; they are habits of thought that apply wherever nonlinear systems are modelled. You have not merely learned about chaos – you have learned to think about nonlinear systems, and that is a tool for the whole of modern science and engineering.

8.6 Exercises

8.6.1 The predictability horizon

Consider the Lorenz equations with the standard parameters $\sigma = 10$, $r = 28$, $b = 8/3$.

- Starting from a point on the attractor, integrate an ensemble of $M = 400$ initial conditions drawn from a small Gaussian blob of width δ_0 . Measure the ensemble spread $\delta(t)$ and verify that it grows as $e^{\Lambda_e t}$ with $\Lambda_e \approx 0.9$, as in Figure 8.3.
- Halve the initial spread δ_0 and repeat. By how much does the predictability horizon increase? Explain, using the relation $t_h \approx \Lambda_e^{-1} \log(\delta_{\text{att}}/\delta_0)$, why improving the initial condition buys only a logarithmically longer forecast.
- At what value of $\delta(t)$ does the exponential growth cease? Relate this saturation level to the size of the attractor.

8.6.2 A Kalman filter for the Lorenz system

Reproduce the twin experiment of Figure 8.4.

- Integrate a "truth" trajectory and generate noisy observations of the full state at intervals $\Delta t_{\text{obs}} = 0.2$, adding Gaussian noise of standard deviation 1.5 to each component. Starting from a deliberately wrong initial guess, run a free forecast (no assimilation) and confirm that it diverges from the truth within the predictability horizon.
- Implement the extended Kalman filter of section 8.2, reusing the Jacobian J of chapter 6 to propagate the covariance. Show that the analysis tracks the truth and that its RMS error stays well below that of the free forecast.
- Study the effect of the observation frequency and noise. What happens as Δt_{obs} is increased towards the predictability horizon? What happens if only the x -component is observed instead of the full state?

8.6.3 Reduced-order structure

- Build a synthetic field $u(x, t) = \sum_{k=1}^3 a_k(t) \phi_k(x)$ from three fixed spatial patterns $\phi_k(x)$ whose amplitudes $a_k(t)$ are the three components of a Lorenz trajectory, and add a little noise. Form the snapshot matrix and compute its singular value decomposition.
- Plot the singular values. How many modes are needed to capture 99% of the energy, and how does this relate to the number of patterns you put in?
- Reconstruct a single snapshot using one, two and three modes and compare with the full field, as in Figure 8.6. What does the part that the leading modes fail to capture correspond to?

8.6.4 Controlling the logistic map

Consider the logistic map $x_{n+1} = r x_n(1 - x_n)$ at $r = 3.9$, where the dynamics are chaotic.

- a) Locate the unstable fixed point $x^* = 1 - 1/r$ and confirm, from the derivative of the map, that it is unstable.
- b) Implement the OGY idea of section 8.4: at each step, if the state is close enough to x^* , choose the small parameter change δr_n that would map x_n exactly onto x^* , applying it only when $|\delta r_n|$ is below a chosen bound. Show that the orbit is captured onto x^* and held there, as in Figure 8.9.
- c) How does the time taken to capture the orbit depend on the size of the allowed perturbation? What happens if the bound is made very small?

8.6.5 Reconstructing an attractor from one signal

- a) Integrate the Lorenz system and keep only the $x(t)$ time series. Form the delay-coordinate vectors $(x_t, x_{t+\tau}, x_{t+2\tau})$ and plot the reconstructed attractor for a few values of the delay τ , as in Figure 8.10.
- b) Estimate the correlation dimension (section 7.3) of the reconstructed attractor and compare it with the value obtained from the full three-dimensional trajectory. Are they consistent, as Takens' theorem requires?

9 About

From Stability to Chaos began as the lecture notes for *Chaotic Processes* (AP 3011 G), a course at Delft University of Technology. Harmen J. Jonker led the course, and the two authors developed it together while Maarten van Reeuwijk was a PhD student working on turbulent convection in Harmen's group in the physics department.

From the outset the course was built around the idea of learning by doing. Each week the students would work through one exercise in the computer lab, building hands-on intuition for this rich and often surprising field. The exercises gathered in this book are therefore central to it, not an afterthought — they are how the subject is meant to be learned.

We had long intended to turn the notes into a book. It has taken us nearly twenty years to reach this point, and we are delighted that it is finally available.



Harmen J. Jonker is Professor of Atmospheric Physics in the Department of Geoscience and Remote Sensing at [Delft University of Technology](#). He co-founded [Whiffle](#), where he is Chief Innovation Officer. His research focuses on turbulence, atmospheric boundary-layer physics and large-eddy simulation, with applications to weather and wind energy.



Maarten van Reeuwijk is Professor of Urban Fluid Mechanics in the Department of Civil and Environmental Engineering at [Imperial College London](#). His research centres on turbulent transport and buoyancy-driven flows — from jets and plumes to environmental and urban fluid mechanics.

A From the three-body problem to the planar restricted system

In the section on celestial uncertainties (1.2) we studied the motion of a light satellite in the gravitational field of the earth and the moon. There we simply stated the four-dimensional set of equations that was integrated numerically. The purpose of this appendix is to show, step by step, how that compact system follows from Newton's law of gravitation together with three physically motivated simplifications. Along the way we take care to define every quantity that appears.

A.1 The full three-body problem

Consider three point masses m_1 , m_2 and m_3 at time-dependent positions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ and $\mathbf{x}_3(t)$. Each position is a vector in three-dimensional space, so the complete state of the system is described by the three position vectors together with the three velocity vectors $\dot{\mathbf{x}}_i = d\mathbf{x}_i/dt$. As each of these six vectors has three components, the phase space of the full problem has dimension 18.

The only force acting is mutual gravitation. Newton's law of gravitation states that the attractive force exerted on body i by body j has magnitude

$$f_{ij} = G \frac{m_i m_j}{|\mathbf{x}_j - \mathbf{x}_i|^2},$$

where $G = 6.67 \cdot 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$ is the universal gravitational constant and $|\mathbf{x}_j - \mathbf{x}_i|$ is the distance between the two bodies. The force points from body i towards body j , i.e. along the unit vector $(\mathbf{x}_j - \mathbf{x}_i)/|\mathbf{x}_j - \mathbf{x}_i|$. Multiplying the magnitude by this unit vector and summing the contributions of the other two bodies, Newton's second law (mass $\times$ acceleration = force) gives the equations of motion

$$\begin{aligned} m_1 \ddot{\mathbf{x}}_1 &= G m_1 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3} + G m_1 m_3 \frac{\mathbf{x}_3 - \mathbf{x}_1}{|\mathbf{x}_3 - \mathbf{x}_1|^3} \\ m_2 \ddot{\mathbf{x}}_2 &= G m_2 m_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} + G m_2 m_3 \frac{\mathbf{x}_3 - \mathbf{x}_2}{|\mathbf{x}_3 - \mathbf{x}_2|^3} \\ m_3 \ddot{\mathbf{x}}_3 &= G m_3 m_1 \frac{\mathbf{x}_1 - \mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_3|^3} + G m_3 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_3}{|\mathbf{x}_2 - \mathbf{x}_3|^3}. \end{aligned} \tag{A.1}$$

Here the cube in the denominator arises because the $1/r^2$ magnitude is multiplied by the unit vector $(\mathbf{x}_j - \mathbf{x}_i)/|\mathbf{x}_j - \mathbf{x}_i|$, which contributes one more power of the distance. It is this coupled, nonlinear system of second-order equations that Poincaré proved cannot be solved in closed form in general.

A.2 Simplification 1: a negligible satellite mass

We now specialise to the earth–moon–satellite setting. The satellite (m_3) is extraordinarily light compared with the earth (m_1) and the moon (m_2); for the values in Table 1.1 the mass ratio is $m_1 : m_2 : m_3 = 100 : 1 : 10^{-19}$. Every term in (A.1) that describes the pull of the satellite on a heavy body is proportional to m_3 and is therefore utterly negligible. Dropping those terms, the equations for the two heavy bodies (the *primaries*) decouple completely from the satellite:

$$\begin{aligned} m_1 \ddot{\mathbf{x}}_1 &= Gm_1 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3} \\ m_2 \ddot{\mathbf{x}}_2 &= Gm_2 m_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \end{aligned} \quad (\text{A.2})$$

while the satellite still feels both of them,

$$\ddot{\mathbf{x}}_3 = Gm_1 \frac{\mathbf{x}_1 - \mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_3|^3} + Gm_2 \frac{\mathbf{x}_2 - \mathbf{x}_3}{|\mathbf{x}_2 - \mathbf{x}_3|^3}. \quad (\text{A.3})$$

The problem has thus split into two parts: a self-contained two-body problem for the primaries, (A.2)–(A.2), and the motion of the satellite, (A.3), which is driven by the (now known) motion of the primaries. This is why the setting is called the *restricted* three-body problem.

A.3 Simplification 2: solving the two-body problem

The classical way to solve a two-body problem is to change to two new sets of coordinates: the *relative position* of the primaries,

$$\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1,$$

which is the vector pointing from the earth to the moon, and the *centre of mass*,

$$\mathbf{r}_c = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2},$$

which is the mass-weighted average position of the two primaries. Adding (A.2) and (A.2) makes the right-hand sides cancel, while subtracting them (after dividing each by its mass) combines the two attractions into a single term. The result is a pair of decoupled equations,

$$\begin{aligned} \ddot{\mathbf{r}}_c &= 0, \\ \ddot{\mathbf{r}} &= -G(m_1 + m_2) \frac{\mathbf{r}}{|\mathbf{r}|^3}. \end{aligned} \quad (\text{A.4})$$

(A.4) states that the centre of mass is not accelerated at all: it moves in a straight line at constant velocity. We are free to observe the system from a frame of reference that moves along with the centre of mass, and in that frame we may set $\mathbf{r}_c = \dot{\mathbf{r}}_c = \mathbf{0}$ without any loss of generality. (A.4) then describes a single fictitious body at position $\mathbf{r}$ attracted to the origin — exactly the mathematical form of a one-body central-force problem.

The bounded and unbounded solutions of (A.4) are the well-known conic sections: circles, ellipses, parabolas and hyperbolas. We do not need their general form here. What matters is that once the relative motion $\mathbf{r}(t)$ is known, the individual orbits of the two primaries follow directly from the definitions of $\mathbf{r}$ and $\mathbf{r}_c$ (with $\mathbf{r}_c = \mathbf{0}$):

$$\mathbf{x}_1(t) = -\frac{m_2}{m_1 + m_2} \mathbf{r}(t), \quad \mathbf{x}_2(t) = \frac{m_1}{m_1 + m_2} \mathbf{r}(t). \quad (\text{A.5})$$

Each primary therefore traces out a scaled copy of the relative orbit, on opposite sides of the common centre of mass, with the lighter body (the moon) swinging out over the larger distance.

A.4 Simplification 3: circular primary orbits in a plane

Of all the conic-section solutions we choose the simplest, a *circular* orbit of the primaries about their centre of mass,

$$\mathbf{r}(t) = R \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \quad \omega = \sqrt{\frac{G(m_1 + m_2)}{R^3}}, \quad (\text{A.6})$$

where $R = |\mathbf{r}|$ is the (constant) earth–moon distance and ω is the angular frequency of the orbit. Substituting (A.6) into (A.4) confirms that this is indeed a solution: the centripetal acceleration $\omega^2 R$ of the circular motion is exactly balanced by the gravitational attraction $G(m_1 + m_2)/R^2$, which is the origin of the expression for ω .

The final simplification concerns the satellite. If the satellite starts in the orbital plane of the two primaries and its initial velocity has no component out of that plane, then the gravitational pull it experiences always lies in the plane, and its trajectory can never leave it. We may therefore describe the satellite by two coordinates only. Dropping the subscript 3 and writing $\mathbf{x}_3(t) = (x(t), y(t))$ for the satellite position, and introducing its velocity components $u = \dot{x}$ and $v = \dot{y}$, the second-order equation (A.3) becomes the first-order, four-dimensional system

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= Gm_1 [x_1 - x]/d_1^3 + Gm_2 [x_2 - x]/d_2^3 \\ \dot{v} &= Gm_1 [y_1 - y]/d_1^3 + Gm_2 [y_2 - y]/d_2^3, \end{aligned} \quad (\text{A.7})$$

in which d_1 and d_2 are the instantaneous distances from the satellite to the earth and to the moon,

$$d_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}, \quad d_2 = \sqrt{(x_2 - x)^2 + (y_2 - y)^2}. \quad (\text{A.8})$$

The primary positions (x_1, y_1) and (x_2, y_2) are prescribed by the circular solution (A.6) together with (A.5):

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = -\frac{m_2 R}{m_1 + m_2} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{m_1 R}{m_1 + m_2} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}. \quad (\text{A.9})$$

Because these positions depend explicitly on time, the satellite moves in a time-dependent force field; it is precisely this time dependence that opens the door to the chaotic behaviour observed in 1.2.

Equations (A.7)–(A.9) are the compact, four-dimensional model quoted in the main text. Although the order of the system has been reduced from eighteen to four, it still cannot be solved analytically, and we resort to numerical integration to explore its rich dynamics.

B The physics behind the Lorenz equations

The Lorenz equations studied in 6 were simply stated there as a celebrated example of a chaotic flow. The purpose of this appendix is to show where they come from: far from being an abstract construction, they are a drastically truncated model of thermal convection. Following the derivation also gives a physical meaning to the three variables x , y , z and to the parameters σ , r and b .

The Hungarian mathematician George Polya (1887–1985) once said: “If you can't solve a problem, then there is an easier problem you can solve: find it.” In this spirit, Ed Lorenz initiated the revolution of chaos theory with his seminal paper titled “Deterministic Nonperiodic Flow” [Lorenz, 1963]. Lorenz was interested in the predictability of weather forecasting. Instead of attempting to solve the immensely complex system of three-dimensional partial differential equations governing the atmosphere, he stripped away all detail until he was left with only three ordinary differential equations, which now carry his name.

B.1 Rayleigh–Bénard convection

The model Lorenz considered was a layer of air confined between two flat plates, heated from below and cooled from above. This flow is known as *Rayleigh–Bénard convection*. It may seem a far stretch from weather prediction, but it shares some essential features. The heating from below resembles the heating of the earth's surface by solar radiation during the day. As hot air is lighter than cold air, it rises due to its buoyancy, forming *thermals* – pockets of rising hot air – that birds and glider pilots gratefully use to stay airborne for extended periods. In the atmosphere, thermals transport not only heat but also water vapour to the free troposphere, making them of primary importance to the hydrological cycle and thus to weather prediction. Rayleigh–Bénard convection has these thermals too (Figure B.1): if there is a predictability problem in this simplified system, one can be certain that it will also be present in the far more complex atmosphere.

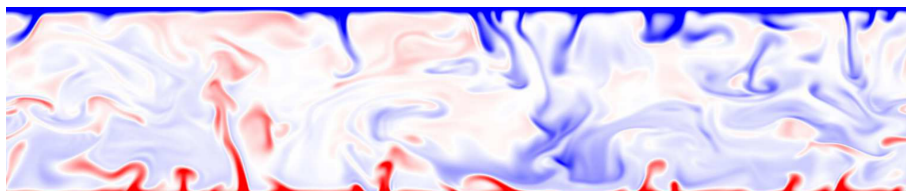


Figure B.1: A snapshot of the temperature field of Rayleigh–Bénard convection obtained by direct numerical simulation, in which the turbulence is resolved down to the smallest scales.

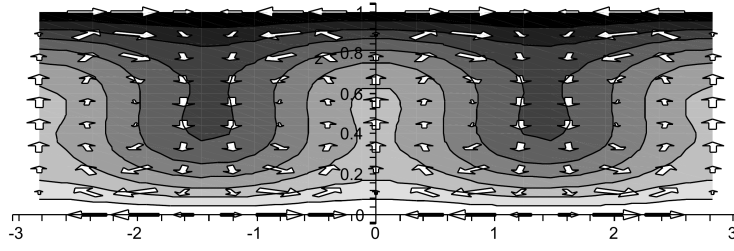


Figure B.2: The velocity and temperature field that the variables x , y and z describe.



Rayleigh–Bénard convection

Under the Boussinesq approximation, the velocity field $\mathbf{u}$, pressure p and temperature deviation Θ of the convecting layer obey conservation of mass, momentum and energy,

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} &= \beta \mathbf{g} \Theta, \\ \frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta - \kappa \nabla^2 \Theta &= 0,\end{aligned}$$

where ρ is the density, ν the kinematic viscosity, κ the thermal diffusivity, β the thermal expansion coefficient and $\mathbf{g}$ gravity. Resolving these equations down to the smallest scales – a direct numerical simulation (Figure B.1) – is enormously demanding [Hanjalić, 2005, Heslot *et al.*, 1987, van Reeuwijk *et al.*, 2006]; the Lorenz equations arise from the opposite extreme, retaining only the very largest scales of motion.

B.2 From convection to three equations

Starting from the convection equations above, Lorenz considered the two-dimensional instability and expressed the fields with a Galerkin method [Holmes *et al.*, 1996, Saltzman, 1962] as an infinite sum of Fourier modes. Retaining only the lowest-order modes leaves three time-dependent amplitudes $x(t)$, $y(t)$ and $z(t)$ that nonetheless characterise an entire velocity and temperature field,

$$\begin{aligned}u(x, z, t) &= -x(t) \frac{(1 + a^2)\pi}{a} \sin(\pi a x) \cos(\pi z), \\ w(x, z, t) &= x(t) \pi(1 + a^2) \cos(\pi a x) \sin(\pi z), \\ \Theta(x, z, t) &= \frac{1}{r\pi} \left(y(t) \cos(\pi a x) \sin(\pi z) - \frac{1}{2} z(t) \sin(2\pi z) \right) - \left(z - \frac{1}{2} \right).\end{aligned}\tag{B.1}$$

The key point is that these expressions are *factorised*: all time dependence resides in x , y and z , while the spatial dependence is carried entirely by the trigonometric terms. Physically, $x(t)$ controls the strength of the convective velocity field; $y(t)$ represents the temperature difference between ascending and descending air; and $z(t)$ measures the deviation of the temperature profile from the linear conduction profile $\Theta(z) = -(z - \frac{1}{2})$. A representative flow field described by ((B.1)–(B.1)) is shown in Figure B.2.

Substituting ((B.1)–(B.1)) into the momentum and temperature equations and neglecting the interactions with the higher-order modes that were dropped, these three amplitudes are found to

obey the closed system

$$\begin{aligned}\dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz.\end{aligned}$$

The parameter r is a non-dimensional temperature difference which drives the flow; the Prandtl number σ is the ratio of kinematic viscosity to thermal diffusivity; and b is a measure of the aspect ratio of the convection rolls, a large value corresponding to a low aspect ratio and vice versa. It should be stressed that this truncation describes only the onset of instability in Rayleigh–Bénard convection, and is quantitatively faithful only near the convective threshold. Remarkably, however, the very same equations describe other physical systems *exactly*, among them a mechanical waterwheel [Strogatz, 1994] as well as certain lasers and dynamos.

It was in this system that Lorenz encountered what seemed, in the early sixties, a genuine paradox. The term *deterministic* was then associated with systems whose long-term fate was to settle onto a fixed point or a limit cycle, whereas *nonperiodic* behaviour was associated with random forcing. Yet here the trajectories remained confined to a bounded region of phase space while every small volume of initial conditions was flattened towards zero volume, and Lorenz could show that the system possessed neither a stable fixed point nor a stable limit cycle. He was forced to conclude that the flow was at once deterministic and nonperiodic – he had stumbled upon a strange attractor.

C Transformation rules for polar coordinates

The relation between Cartesian coordinates and polar coordinates are

$$x = r \cos(\theta), \quad y = r \sin(\theta) \quad (\text{C.1})$$

Differentiating both sides with respect to time results in

$$\begin{aligned} \dot{x} &= \dot{r} \cos(\theta) - r \sin(\theta) \dot{\theta} \\ \dot{y} &= \dot{r} \sin(\theta) + r \cos(\theta) \dot{\theta} \end{aligned} \quad (\text{C.2})$$

Now note that

$$\begin{aligned} \sin(\theta) \dot{x} + \cos(\theta) \dot{y} &= \dot{r} \\ \cos(\theta) \dot{y} - \sin(\theta) \dot{x} &= r \dot{\theta} \end{aligned}$$

Substituting $\sin \theta = y/r$ and $\cos(\theta) = x/r$ we obtain

$$\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \end{aligned} \quad (\text{C.3})$$

Equations ((C.3)) are used to transform the ODE to polar coordinates. The inverse transform is defined by ((C.2)).

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